Math 128B, problem set 11
REVISED AND CORRECTED SUN MAY 15
Outline due: Wed May 11
Due: Mon May 16
Last revision due: Wed May 18

Problems to be done, but not turned in: (Ch. 32) 1, 3, 5, 7, 9, 11, 13, 15, 17.

Problems to be turned in:

1. (Ch. 32) 4.
2. Let E be the splitting field of $x^4 + 1$ over $\mathbf{Q}$.
   (a) Find $\mathrm{Gal}(E/\mathbf{Q})$, and list all of its elements explicitly.
   (b) Find all subfields of E, and draw the lattice of subfields of E. For each subfield K, write K in the form $K = \mathbf{Q}(a)$, indicate $[K : \mathbf{Q}]$, and indicate which subgroup of $\mathrm{Gal}(E/\mathbf{Q})$ fixes K.
3. (Ch. 32) 18. Prove your answer.
4. Let F be a field of characteristic 0, and let E be the splitting field of some $f(x) \in F[x]$ such that $\mathrm{Gal}(E/F) \approx S_4$. Prove that E has a subfield K containing F such that $[K : F] = 4$.
5. Let E be the splitting field of some $f(x) \in \mathbf{Q}[x]$ such that $\mathrm{Gal}(E/\mathbf{Q}) \approx A_4$.
   (a) Draw the lattice of subfields of E, in the following sense: Let $K_1, K_2, \ldots$ be the subfields of E. Draw as much of the subfield lattice of E as you can derive from the given information. Also, for each subfield K_i, indicate $[K_i : \mathbf{Q}]$, and indicate which subgroup of $\mathrm{Gal}(E/\mathbf{Q})$ fixes K_i.
   (b) Which subfields of E are splitting fields of some $g(x) \in \mathbf{Q}[x]$? Justify your answer.
6. Let F be a field of characteristic 0, and let E be an extension of F such that $[E : F] = 935 = 5 \cdot 11 \cdot 17$.
   (a) Prove that there exists a polynomial $f(x) \in F[x]$ whose splitting field is isomorphic to some subfield K of E containing F such that $[K : F] = 55$.
   (b) Prove that there exists a subfield L of E containing F such that $[L : F] = 5$.