Math 128B, problem set 06
Outline due: Wed Mar 16
Due: Mon Mar 21
Last revision due: Mon May 02

Problems to be done, but not turned in: (Ch. 19) 3, 9, 11, 13, 17, 19, 25, 29, 31.
Fun: (Ch. 18) 32.

Problems to be turned in:

1. (Ch. 18) 8.
2. (Ch. 18) 22.
3. Let D be a principal ideal domain. For $p \in D$, prove that the following are equivalent:
   (a) The ideal $\langle p \rangle$ is maximal.
   (b) The ideal $\langle p \rangle$ is prime.
   (c) The element p is irreducible.

   Suggestion: Prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). Which of these implications still work if D is just an integral domain and not necessarily a PID? Does the equivalence still hold if D is a UFD and not necessarily a PID? Prove or give a counterexample.
4. Let D be a principal ideal domain, and let a, b be nonzero elements of D. Prove that there exists some $g \in D$ (depending on a and b) such that:
   - The element g divides both a and b;
   - For $d \in D$, if d divides both a and b, then d divides g; and
   - There exist $x, y \in D$ such that $ax + by = g$.

   (Note: Because of the first two properties, we may think of g as $\gcd(a, b)$, the greatest common divisor of a and b.)

   Suggestion: Consider $\langle a \rangle + \langle b \rangle$, which we know is an ideal of D (Ch. 14 #10).
5. Let F be a field, and suppose that for $p(x), q(x) \in F[x]$, we have that $\gcd(p(x), q(x)) = 1$ (see problem 4). Prove that for any $f(x) \in F[x]$, there exist $r(x), s(x) \in F[x]$ such that
$$\frac{f(x)}{p(x)q(x)} = \frac{r(x)}{p(x)} + \frac{s(x)}{q(x)}.$$
   (Note: Suitably refined, this is the theorem that makes the integration technique of partial fractions possible.)
6. (Ch. 19) 22. Prove your answer.
7. Let V be a vector space over the field F.
   (a) Prove that if $\{v_1, \ldots, v_k\} \subseteq V$ is linearly independent, and $x \in V$ is not a linear combination of $v_1, \ldots, v_k$, then $\{v_1, \ldots, v_k, x\}$ is also linearly independent.

(b) Prove that if $\{w_1, \ldots, w_n\}$ spans V, and $\{v_1, \ldots, v_k\}$ is linearly independent, then the union of $\{v_1, \ldots, v_k\}$ and some subset of $\{w_1, \ldots, w_n\}$ is a basis for V.

(c) Prove that if $\dim V = n$, and $\{v_1, \ldots, v_k\}$ is linearly independent, then $k \leq n$.