Math 128B, problem set 05
CORRECTED THU MAR 03
Outline due: Wed Mar 09
Due: Mon Mar 14
Last revision due: Mon Apr 04

Problems to be done, but not turned in: (Ch. 17) 17, 21, 23, 25, 27, 29, 31, 33. (Ch. 18) 1, 3, 5, 7, 13, 15, 19, 23, 27, 29, 37.

Fun: (Ch. 18) 33.

Problems to be turned in:

1. Let p be a prime, let $f(x)$ be a polynomial with integer coefficients, and let $\overline{f}(x)$ be the polynomial in $\mathbf{Z}_p[x]$ obtained by reducing the coefficients of $f(x)$ mod p.

   (a) Consider the following statements.

      i. $f(x)$ is reducible over $\mathbf{Z}$.

      ii. $f(x)$ is reducible over $\mathbf{Q}$.

      iii. $\overline{f}(x)$ is reducible over $\mathbf{Z}_p[x]$.

      There are six possible implications between two statements of (i)–(iii). For each possible implication ((i) implies (ii), (ii) implies (i), etc.), either give a proof, give a reference to a result proven in the text, or give a counterexample.

   (b) If we assume that $f(x)$ is monic, what changes in part (a)? Explain.

2. (Ch. 17) 8.

3. (Ch. 17) 10.

4. (Ch. 17) 18.

5. Let F be a field.

   (a) By completing the square, re-derive (with proof) the quadratic formula for finding the zeros of a polynomial ax^2+bx+c. For which fields F does this formula work?

   (b) Now assuming that F is a field in which the quadratic formula works, use the quadratic formula to obtain a criterion for the irreducibility of $ax^2 + bx + c$, in terms of squares.

6. Since $2\mathbf{Z}$ is a ring without unity, for $a \in 2\mathbf{Z}$, we say that a is *reducible* in $2\mathbf{Z}$ if $a = bc$ for some $b, c \in 2\mathbf{Z}$; otherwise, we say that a is *irreducible* in $2\mathbf{Z}$.

   (a) List the irreducible elements of $2\mathbf{Z}$. (Prove/explain your answer.)

   (b) Find some $n \in 2\mathbf{Z}$ such that $n = ab = cd$ and $a, b, c, d > 0$ are all distinct irreducible elements of $2\mathbf{Z}$.

7. (Ch. 18) 2.