Math 128B, Wed May 05

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: Review Chs. 1, 4, 5, 7, 9, 10. (S_n, A_n, D_n, $C_n \approx \mathbf{Z}_n$); new reading pp. 387–388.
- Reading for next Mon: Ch. 32.
- PS10 outline due Fri May 07.

$V = C_2 \oplus C_2$

$\approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$

C_n b/c mult

C_n is my notation for the multiplicative cyclic group of order n.

C_n = <a>, where a has order n.

F* is the group of nonzero elements of field F, aka multiplicative group of F, which is cyclic if F finite.

GF(p^e) is the finite field of order p^e, p prime, e >= 1.
GF(p^e)* is cyclic of order p^e-1 (b/c GF(p^e) has p^e-1 nonzero elements).

So GF(p^e)* isom to C_{p^e-1}.

Z_4 = integers mod 4 under addition
 1, 3 generate Z_4; 0, 2 don't.

C_4 = cyclic group of order 4, operation multiplication, generated by some a with order 4 (so a^4 = e). a will not usually be an integer.
 a^1, a^3 generate C_4; a^0=e, a^2 don't.

C_{10} = cyclic of order 10, operation mult.
 a^2, a^4, a^6, a^8, a^5, a^0=e don't generate C_{10}
 all other elements (a^1, a^3, a^7, a^9) of C_{10} generate C_{10}.

Recap: Conjugacy and the cycle-shape theorem

Note: Conjugacy is an equivlance relation, so ccs are equivalence classes under conjugacy.

Definition

G a group. To say that $a \in G$ is **conjugate** to $b \in G$ means that there exists some $g \in G$ such that $gag^{-1} = b$. The **conjugacy class** of $a \in G$ is the set of all elements of G conjugate to a, i.e.,

$$\left\{gag^{-1} \mid g \in G\right\}.$$

Theorem (Cycle-shape)

For $\alpha, \sigma \in S_n$, let $\beta = \sigma\alpha\sigma^{-1}$. Then β has the same cycle-shape as α, except renumbered by σ; that is, conjugation by σ turns each cycle of α of the form $(a\ b\ c\ \ldots\ z)$ to a cycle of the form $(\sigma(a)\ \sigma(b)\ \sigma(c)\ \ldots\ \sigma(z))$.
Conversely, for $\alpha, \beta \in S_n$ with the same cycle-shape, there exists some $\sigma \in S_n$ such that $\beta = \sigma\alpha\sigma^{-1}$.

$g = (1\ 3\ 4)$ $a = (1\ 2)(3\ 6)$

Thm: $gag^{-1} = (3\ 2)(4\ 6)$

$= (2\ 3)(4\ 6)$

To review why normal subgroups are important/useful and what conjugacy has to do with a subgroup being normal: See Ch. 9.

Q: Is there a group G such that all non-identity elements are conjugate?
Silly answer: C_2.
Non-silly answer: Yes, but G is infinite and is constructed in a fancy recursive (IIRC).

D_4, C_4, and $V \approx C_2 \oplus C_2$ (Chs. 1, 4)

$D_4 = \{\underbrace{e, (1234), (13)(24), (1432)}_{\text{rotns}}, (24), (13), (12)(34), (14)(23)\}$

$C_4 \leq D_4$, $C_4 = \{e, \underset{\substack{\parallel \\ a}}{(1234)}, (13)(24), (1432)\} = \langle a \rangle$

$C_2 \oplus C_2$
$\cong$
$V = \{e, (13)(24), (12)(34), (14)(23)\}$

see PS01 128A

Recall: Every group of order 4 is isomorphic either to C_4 or $C_2 \times C_2$ (Gallian Ch. 8, though really in Ch. 7).

More generally: Every group of order 2p, where p is prime, is isomorphic either to C_{2p} or D_p (symmetries of regular p-gon).

Fact S_4, A_4, D_4, C_4, V are the transitive perm gps on 4 objects.

Ex V transitive b/c:

$e: 1 \to 1$

$(12)(34): 1 \to 2$

$(13)(24): 1 \to 3$

$(14)(23): 1 \to 4$

$S_3 \approx D_3$ (Chs. 1, 5) and $C_3 \approx A_3$ (Chs. 4, 5)

Shapes of elements, numbers of elements of each type.

$$S_3 = \{ e, \underset{3}{(a\,b)}, \underset{2}{(a\,b\,c)} \}$$

$$= \{ e, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2) \}$$

$$A_3 = \{ e, (1\,2\,3), (1\,3\,2) \} \approx C_3$$

Important subtlety: (1 2 3) and (1 3 2) are conjugate in S_3, but they are *not* conjugate in A_3.

$$\underset{g}{(23)}\underset{a}{(1\,2\,3)}\underset{g^{-1}}{(2\,3)} = \underset{b}{(1\,3\,2)}$$

But $g \in S_3$, not in A_3

On the other hand, A_3 is abelian, and in an abelian group, every element is conjugate only to itself.

Ab $gag^{-1} = gg^{-1}a = a$

Subgroups of A_4

Recall: $|A_4| = 12$, elements are:

$\cancel{4}$

$3+1$ $(a\,b\,c)$

$(123)(132)$
$(124)(142)$
$(134)(143)$
$(234)(243)$

a, a^{-1}

a, a^2

$2+2$ $(a\,b)(c\,d)$ $= (12)(34), (13)(24), (14)(23)$

$\cancel{2+1+1}$

$1+1+1+1$ e

Cyclic subgroups:

Order 2: $\langle (12)(34) \rangle$ $\langle (13)(24) \rangle$ $\langle (14)(23) \rangle$

order 3:

$\langle (123) \rangle, \langle (124) \rangle, \langle (134) \rangle, \langle (234) \rangle$

Subgroups of A_4, cont.

$1, 2 ✓, 3 ✓, 4 ✓, \cancel{6}, 12 ✓$

Subgroups of orders 4 and 6?

ord 4: V is only poss.

No ord 6: If $\exists \in H$ s order 2, 3

e.g. (123), $(12)(34)$

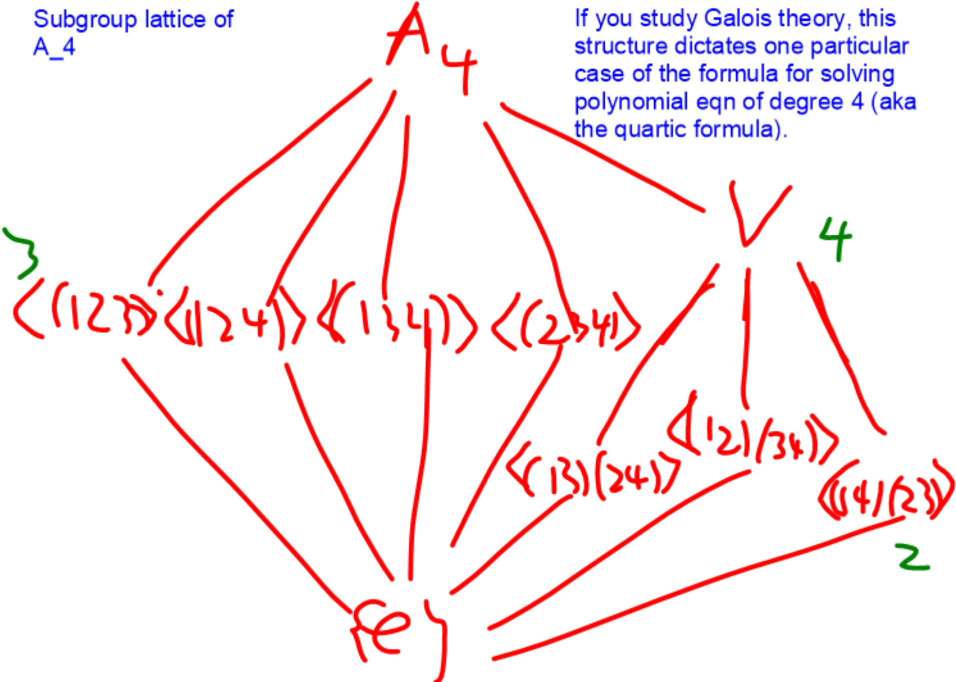
Conj $(12)(34) \xrightarrow{(123)} (23)(14) \xrightarrow{(123)} (31)(24)$

So $V \subseteq$ subgr, so 4 div $|H|$

$\Rightarrow H = A_4$

Subgroup lattice of A_4

If you study Galois theory, this structure dictates one particular case of the formula for solving polynomial eqn of degree 4 (aka the quartic formula).



The Orbit-Stabilizer Theorem and conjugacy

Suppose G permutes a set S (for example, by conjugation). For $i \in S$, define

stabilizer of i in G: $\mathrm{stab}_G(i) = \{\alpha \in G \mid \alpha(i) = i\}$,

orbit of i under G: $\mathrm{orb}_G(i) = \{\alpha(i) \mid \alpha \in G\}$.

G is a group permuting its own elements by conjugation.

Theorem (Orbit-Stabilizer) Ch. 7

For $i \in S$, $|G| = |\mathrm{orb}_G(i)|\,|\mathrm{stab}_G(i)|$.

Exmp. S_5 permuting the conjugacy classes of ~~(1 2)(3 4)~~ (1 3)(2 4), (1 2 3), (1 2 3 4 5):

15 elts $(a\,b)(c\,d)$, $|S_5| = 5! = 120$

$|\mathrm{stab}_{S_5}((a\,b)(c\,d))| = \frac{120}{15} = 8$

So: Stab of $(13)(24)$ under conj has order 8

Turns out:

$$\mathrm{stab}_{S_5}((13)(24)) = D_4$$

$$\parallel$$

$$\mathrm{Cent}_{S_5}((13)(24))$$

The conjugacy classes of A_5

Normal subgroups and simple groups

Definition

Let $H \leq G$. To say that H is **normal** means that for any $a \in H$ and $g \in G$, we have that $gag^{-1} \in H$. (Note that even if $gag^{-1} \in H$, it need not be the case that $gag^{-1} = a$.) In that case, we write $H \lhd G$.

Note that a subgroup $H \lhd G$ exactly when H is a union of conjugacy classes.

Definition

To say that a group G is **simple** means that the only normal subgroups of G are $\{e\}$ and G.

A_5 is simple