Math 128B, Wed Apr 28

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today and next week: Review Chs. 1, 4, 5, 7, 9, 10. (S_n, A_n, D_n, $C_n \approx \mathbf{Z}_n$); new reading pp. 387–388.
- PS09 due tonight. Or by next week: Class notes.
- Exam 3, Mon May 03.
- Exam review Fri Apr 30, 10am–noon.

128B: Probably 11-noon

The Minimal Polynomial Theorem

Let E be an extension of a field F, let $a \in E$ be algebraic over F, and suppose $m(x) \in F[x]$ is monic.
Then the following are equivalent:

1. $m(x)$ is irreducible over F and $m(a) = 0$.
2. $m(x)$ is a nonzero polynomial of smallest possible degree such that $m(a) = 0$.
3. $[F(a) : F] = \deg m(x)$ and $m(a) = 0$.
4. $F(a) \approx F[x]/\langle m(x) \rangle$ and $m(a) = 0$.

Furthermore, if any (and therefore all) of the above conditions hold, then for any $f(x) \in F[x]$ such that $f(a) = 0$, we have that $m(x)$ divides $f(x)$ in $F[x]$.

Thm 21.3

Review (Ch. 5): Permutation groups

Definition

S_n is the group of all permutations on n objects.
A_n is the subgroup of S_n consisting of all **even** permutations on n objects. (Cycles of odd length are even perms, and vice versa.)
A **permutation group** on n objects is a subgroup of S_n.

Definition

To say that a permutation group G on n objects is **transitive** means that for any $a, b \in \{1, \ldots, n\}$, there is some $\alpha \in G$ such that $\alpha(a) = b$. ("You can always get here from there.")

To prove that the quintic is unsolvable:

- Need to understand transitive permutation groups on 4 and 5 objects. also 2 and 3 but those are less complicated
- Need to understand all subgroups of those groups, especially normal vs. non-normal subgroups.

Conjugacy (Ch. 24, new)

Definition

G a group. To say that $a \in G$ is **conjugate** to $b \in G$ means that there exists some $g \in G$ such that $gag^{-1} = b$. The **conjugacy class** of $a \in G$ is the set of all elements of G conjugate to a, i.e.,

$$\left\{ gag^{-1} \mid g \in G \right\}.$$

Note that a subgroup is **normal** exactly when it is also closed under conjugacy.

Example: $a \in S_6$, random examples of $g \in S_6$:

$g = (1\,2\,3) \quad a = \underbrace{(1\,5\,4\,3\,6)}$ 5-cycle

$gag^{-1} = (1\,2\,3)(1\,5\,4\,3\,6)(1\,3\,2)$

$= \overbrace{(1\,6\,2\,5\,4)}$ 5-cycle

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 3 & 4 & 2 \end{pmatrix} \qquad \alpha = \underbrace{(1\,3\,5\,6\,4\,2)}_{6\text{-cycle}}$$

$$= (2\,6)(3\,5\,4)$$

$$1 \leftarrow 1 \leftarrow 2 \leftarrow 6$$

$$\sigma\alpha\sigma^{-1} = (2\,6)(3\,5\,4)(1\,3\,5\,6\,4\,2)(2\,6)(3\,4\,5)$$

$$= \underbrace{(1\,5\,4\,2\,3\,6)}_{6\text{-cycle}}$$

The cycle-shape theorem

Theorem

For $\alpha, \sigma \in S_n$, let $\beta = \sigma\alpha\sigma^{-1}$. Then β has the same cycle-shape as α, except renumbered by σ; that is, conjugation by σ turns each cycle of α of the form

$$(a\ b\ c\ \ldots\ z)$$

to a cycle of the form

$$(\sigma(a)\ \sigma(b)\ \sigma(c)\ \ldots\ \sigma(z)).$$

Consequently, for $\alpha, \beta \in S_n$ there exists some $\sigma \in S_n$ such that $\beta = \sigma\alpha\sigma^{-1}$.

$\boxed{\text{if same cycle shape}}$

"Proof" by example.

$\sigma = (1\,2\,3)$

$\alpha = (1\,5\,4\,3\,6)$

__Thm:__ $\sigma\alpha\sigma^{-1} =$

__says__

$$(\sigma(1)\ \sigma(5)\ \sigma(4)\ \sigma(3)\ \sigma(6))$$
$$= (2\ 5\ 4\ 1\ 6)$$
$$= (1\ 6\ 2\ 5\ 4)$$

$$\alpha = (1\ 3)(4\ 2\ 5)$$
$$\beta = (3\ 2)(5\ 4\ 1) = (1\ 5\ 4)(2\ 3)$$
$$\sigma = \begin{pmatrix} 1 & 3 & 4 & 2 & 5 & 6 \\ 3 & 2 & 5 & 4 & 1 & 6 \end{pmatrix} = (1\ 3\ 2\ 4\ 5)$$

sigma chosen precisely to renumber alpha to beta

If $z\alpha = \alpha z$ $\left(z \in C_{S_n}(\alpha)\right)$

$\Rightarrow$ $z\alpha z^{-1} = \alpha$

Centralizer of alpha is exactly the stabilizer of alpha under conjugation (!!!).

qnA $\sigma\alpha\sigma^{-1} = \beta$

Then $(\sigma z)\alpha(\sigma z)^{-1}$

$= \sigma \underbrace{z\alpha z^{-1}} \sigma^{-1}$ socks & shoes

$= \sigma\alpha\sigma^{-1} = \beta$

So σ unique up to cosets of $C_{S_n}(\alpha)$.

S_4 (Ch. 5)

I.e., what are the conjugacy classes in S_4?

$|S_4| = 24$

Shapes of elements, numbers of elements of each type.

A_4

4	$(bcda) = (a\,b\,c\,d)$	$\frac{4!}{4} = 6$	
3+1	$(a\,b\,c)$	$4\frac{3!}{3} = 8$	✓
2+2	$(a\,b)(c\,d)$	$= 3$	✓
2+1+1	$(a\,b)$	$\binom{4}{2} = 6$	
1+1+1+1	e	$= 1$	✓

$(1\,2\,3\,4) = (2\,3\,4\,1) = (3\,4\,1\,2) = (4\,1\,2\,3)$

$(1\,2)(3\,4) \quad (1\,3)(2\,4) \quad (14)(23)$

How to choose (abc) in S_4

1. Pick fixed $d \neq a,b,c$ $\quad 4 = \binom{4}{1} = \binom{4}{3}$

2. Pick 3-cycle using a,b,c $\quad \frac{3!}{3}$ $\quad \frac{3!}{3}$

Total $\quad 4 \cdot \frac{3!}{3}$

$\#k\text{-cycles in } S_n = \binom{n}{k} \cdot \left(\frac{k!}{k}\right)$

A_4 (Ch. 5)

Shapes of elements, list all elements. in conjugacy classes.

Shapes	Elts	Conj in A_4
(abc)	(123) (132) (124) (142) (134) (143) (234) (243)	(123) (243) (142) (134) ; (132) (124) (143) (234)
$(ab)(cd)$	$(12)(34)$, $(13)(24)$, $(14)(23)$	one class in A_4
e	e	one class in A_4

All of these elements are conjugate in S_4, but not necessarily in A_4

subgp V

Important point: If a normal subgroup contains one element of a conjugacy class, it must contain ALL of the elements of that conjugacy class.

Conversely: A subgroup that is a union of conjugacy classes must be normal.

D_4, C_4, and $V \approx C_2 \oplus C_2$ (Chs. 1, 4)

$V = \{e, (12)(34), (13)(24), (14)(23)\}$

$C_4 = \langle (1234) \rangle$

$D_4 =$ symms of

$\{e, (1234), (13)(24), (1432), (14)(23), (12)(34), (24), (13)\}$

V

Solve $x^4 - 7$

$S_3 \approx D_3$ (Chs. 1, 5) and $C_3 \approx A_3$ (Chs. 4, 5)

Shapes of elements, numbers of elements of each type.

Subgroups of A_4

Just saw: $|A_4| = 12$, elements are:

Cyclic subgroups:

Subgroups of A_4, cont.

The Orbit-Stabilizer Theorem and conjugacy

Suppose G permutes a set S. For $i \in S$, define

$$\begin{aligned} \mathrm{stab}_G(i) &= \{\alpha \in G \mid \alpha(i) = i\}, \\ \mathrm{orb}_G(i) &= \{\alpha(i) \mid \alpha \in G\}. \end{aligned}$$

Theorem (Orbit-Stabilizer)

For $i \in S$, $|G| = |\mathrm{orb}_G(i)|\,|\mathrm{stab}_G(i)|$.

Exmp. S_5 permuting the conjugacy classes of (1 2)(3 4), (1 2 3), (1 2 3 4 5)

The conjugacy classes of A_5