This test consists of 7 questions on 2 pages, totalling 100 points. You are allowed to use one page of notes. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

1. (13 points) Let $R = \mathbf{Z} \oplus \mathbf{Z}_3$.

(a) List all units of R. (If the list is infinite, give a precise description of the elements of that list.)

(b) List all zero-divisors of R. (If the list is infinite, give a precise description of the elements of that list.)

No explanation necessary.

For questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (13 points) **TRUE/FALSE:** Define $\varphi : \mathbf{Z}_{30} \to \mathbf{Z}_{30}$ by $\varphi(x) = 3x$ for all $x \in \mathbf{Z}_{30}$. Then φ is a ring homomorphism.

3. (13 points) **TRUE/FALSE:** Let R be a commutative ring with unity, and let A be an ideal of R such that R/A is isomorphic to $\mathbf{Z}$. Then it must be the case that A is a maximal ideal of R.

4. (13 points) **TRUE/FALSE:** It is possible that there exists a commutative ring R with $a, b \in R$ such that $ab = 0$, $a \neq 0$, and $b \neq 0$.

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5. (15 points) **PROOF QUESTION.** Let R be a ring (not necessarily commutative), let T be a commutative ring, and let $\varphi : R \to T$ be a ring homomorphism. Prove that if $r, s \in R$, then $rs - sr \in \ker \varphi$. Justify each step carefully.

6. (15 points) **PROOF QUESTION.** Consider the ideal $A = \langle x^2 - 1 \rangle$ of the ring $R = \mathbf{R}[x]$ (ring of polynomials with real coefficients).

(a) Find a zero-divisor α in the factor ring R/A, and prove that α is a zero-divisor of R/A, justifying each step carefully.

(b) Is R/A is a field? Prove your answer.

Note that if you do not know how to do part (a) of this problem, you may assume the result of part (a) in order to do part (b).

7. (18 points) **PROOF QUESTION.** Let R be a commutative ring, let A and B be ideals of R, and let

$$A + B = \{a + b \mid a \in A,\ b \in B\}.$$

Use the Ideal Test (and not results proved on the homework, etc.) to prove that $A + B$ is an ideal of R.