Group actions
Math 128B
Supplement to Gallian Ch. 24

Definition 1. Let X be a set. We define $\mathrm{Sym}(X)$ to be the group of all bijections $X \to X$, or the *symmetric group on* X. Note that if $X = \{1, \dots, n\}$, $\mathrm{Sym}(X)$ is just the usual symmetric group S_n.

Definition 2. Let G be a group, and let X be a set. We define an *action* of G on X to be a (group) homomorphism $\rho : G \to \mathrm{Sym}(X)$. Note that since elements of $\mathrm{Sym}(X)$ are functions from X to X, instead of writing the image of $g \in G$ as $\rho(g)$, we write its image as ρ_g. In other words, to say that ρ is an action of G on X is to say that for each $g \in G$, we have a bijection $\rho_g : X \to X$ such that for $g, h \in G$, $\rho_{gh} = \rho_g \rho_h$.

Definition 3. Let G be a group, let X be a set, and let $\rho : G \to \mathrm{Sym}(X)$ be an action of G on X. For $x \in X$, we define the *orbit of x under the action of G* to be

$$\mathrm{orb}_G(x) = \{\rho_g(x) \mid g \in G\} = \{y \in X \mid y = \rho_g(x) \text{ for some } g \in G\} \subseteq X, \tag{1}$$

and we define the *stabilizer of x under the action of G* to be

$$\mathrm{stab}_G(x) = \{g \in G \mid \rho_g(x) = x\} \leq G. \tag{2}$$

Example 4. Let G be a permutation group (i.e., a subgroup of S_n) and let $X = \{1, \dots, n\}$. Then the natural embedding $\rho : G \to S_n$ defines an action of G on X, and $\mathrm{orb}_G(x)$ and $\mathrm{stab}_G(x)$ are as defined in Gallian Ch. 7.

Example 5. Let $G = D_n$ and let X be the set of the vertices of a regular n-gon. Then the usual geometric picture of D_n (see Gallian Ch. 1) defines an action of G on X; furthermore, for $x \in X$, $\mathrm{orb}_G(x) = X$ and $\mathrm{stab}_G(x)$ is a cyclic group of order 2 generated by the reflection whose axis of symmetry passes through x. Analogous statements hold if X is the set of edges of a regular n-gon.

Example 6. Let G be any group, and let $X = G$. Then the map $\lambda : G \to \mathrm{Sym}(X)$ defined by $\lambda_g(x) = gx$ defines an action of G on itself by left multiplication. (This is the *left regular representation* of G; see Gallian Thm. 6.1.) Furthermore, for $x \in X = G$, $\mathrm{orb}_G(x) = X = G$ and $\mathrm{stab}_G(x)$ is trivial.

Example 7. Let G be any group, and let $X = G$. Then the map $\varphi : G \to \mathrm{Sym}(X)$ defined by $\varphi_g(x) = gxg^{-1}$ defines an action of G on itself by left conjugation (see Gallian Thm. 24.1). Furthermore, for $x \in X = G$, $\mathrm{orb}_G(x)$ is the *conjugacy class of x*, and $\mathrm{stab}_G(x) = C(x)$, the *centralizer of x*.

Example 8. Let G be any group, and let X be the set of all subgroups of G. Then the $\varphi : G \to \mathrm{Sym}(X)$ defined by $\varphi_g(H) = gHg^{-1}$ defines an action of G on X by left conjugation (see Gallian, proof of Thm. 24.4). Furthermore, for $H \in X$, $\mathrm{orb}_G(H)$ is the *conjugacy class of H*, and $\mathrm{stab}_G(H) = N(H)$, the *normalizer of H*.

Exercise. Verify the above statements about orbits and stabilizers.

Theorem 9. *Let G be a group, let X be a set, and let $\rho : G \to \mathrm{Sym}(X)$ be an action of G on X. For any $x \in X$, let C be the set of all left cosets of the coset $\mathrm{stab}_G(x)$ in G. Then the function $\Phi : C \to \mathrm{orb}_G(x)$ defined by*

$$\Phi(a\,\mathrm{stab}_G(x)) = \rho_a(x) \tag{3}$$

is well-defined and bijective. In particular, if $\mathrm{orb}_G(x)$ is finite, then

$$|\mathrm{orb}_G(x)| = |G : \mathrm{stab}_G(x)|\,. \tag{4}$$

Proof. The only possible ambiguity in the definition of Φ comes in the choice of coset representative a in the coset $a\,\mathrm{stab}_G(x)$. However, if a' is another representative for $a\,\mathrm{stab}_G(x)$, then $a' = ah$ for some $h \in \mathrm{stab}_G(x)$ (Gallian, Lemma in Ch. 7), and

$$\rho_{a'}(x) = \rho_{ah}(x) = \rho_a\rho_h(x) = \rho_a(x), \tag{5}$$

by the definition of $\mathrm{stab}_G(x)$.

To see that Φ is surjective, for $y \in \mathrm{orb}_G(x)$, by definition of orbit, $y = \rho_g(x)$ for some $g \in G$, which means that $y = \Phi(g\,\mathrm{stab}_G(x))$. To see that Φ is injective, suppose $\Phi(a\,\mathrm{stab}_G(x)) = \Phi(b\,\mathrm{stab}_G(x))$. Then $\rho_a(x) = \rho_b(x)$, which means that

$$\rho_{a^{-1}b}(x) = \rho_{a^{-1}}\rho_b(x) = \rho_a^{-1}\rho_a(x) = x. \tag{6}$$

Therefore, $b^{-1}a \in \mathrm{stab}_G(x)$, which means that $a\,\mathrm{stab}_G(x) = b\,\mathrm{stab}_G(x)$ (Gallian, Lemma in Ch. 7). □

Remark 10. Let G be a group, let X be a set, and let $\rho : G \to \mathrm{Sym}(X)$ be an action of G on X. In other classes, the ρ part of the notation is omitted/implied, and an action of G on X is written as a way of defining the "product" $g.x$. In other words, we may define a (left) action of G on X to be a way to define the product $g.x$ such that, for all $x \in X$ and $g_1, g_2 \in G$,

$$1.x = x, \tag{7}$$
$$(g_1g_2).x = g_1.(g_2.x). \tag{8}$$

Exercise. Prove that the definition of action in Remark 10 is equivalent to Definition 2. (What role does (7) play in the definition?)

Remark 11. One advantage to the definition of left action in Remark 10 is that it allows us to define what is meant by a *right action* $x.g$. This is useful primarily because it is sometimes helpful to consider left and right actions simultaneously. In particular, we can denote the fact that certain left and right actions commute as an "associtivity" property, i.e., $(g.x).h = g.(x.h)$.