This test consists of 13 questions on 3 pages, totalling 200 points. You are allowed to use one page of notes. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

**1.** (15 points) In $S_{11}$, let

$$\alpha = (1\ 7)(2\ 3\ 9\ 8\ 6\ 10)(4\ 11\ 5)$$
$$\beta = (1\ 7\ 11\ 10\ 6\ 2\ 4\ 3\ 5)(8\ 9)$$

Calculate $\alpha\beta$, $\alpha^{-1}$, and $\mathrm{ord}(\alpha)$. Put all permutation answers in cycle form, and show all your work.

**2.** (15 points)

(a) How **many** abelian groups of order $300125 = 5^3 7^4$ are there, up to isomorphism? Briefly **justify** your answer. (You do not need to list all of those groups, but you do need to explain where your answer comes from.)

(b) Give **one** example of an abelian group $G$ of order $300125 = 5^3 7^4$ that has two properties:

- $G$ is not cyclic; and
- $G$ has an element of order greater than 35.

No explanation necessary.

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For questions 3–8, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

**3.** (15 points) **TRUE/FALSE:** Let $G$ be a group of (finite) order $n$ and let $H$ be a group of (finite) order $k$. It is possible that $G \oplus H$ has an element of order $nk$.

**4.** (15 points) **TRUE/FALSE:** If $G$ is a group and $a, b \in G$, it must be the case that $(ab)^{-1} = a^{-1}b^{-1}$.

**5.** (15 points) **TRUE/FALSE:** The group $S_4$ is isomorphic to the group $\mathbf{Z}_4 \oplus \mathbf{Z}_6$.

**6.** (15 points) **TRUE/FALSE:** Let $G$ be a group of order 64. If $H$ is a subgroup of $G$ such that $H \neq \{e\}$, then $|H|$ must be even.

**7.** (15 points) **TRUE/FALSE:** Let $G$ be a group and let $N$ be a normal subgroup of G. If $a$ is an element of $G$ of order 6, then it must be the case that $aN$ is an element of $G/N$ of order 6.

**8.** (15 points) **TRUE/FALSE:** Let $R$ be a commutative ring with multiplicative identity 1. For $a \in R$, if $a \neq 0$, then there exists some $b \in R$ such that $ab = 1$.

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**9.** (16 points) **PROOF QUESTION.** Recall that $D_6$ denotes the symmetry group of the regular hexagon. Suppose that $H$ is a subgroup of $D_6$ that contains no reflections (i.e., no reflection is an element of $H$). Prove that $H$ is cyclic.

**10.** (16 points) **PROOF QUESTION.** Let $G$ be a group and let $H$ be a subgroup of $G$. Define

$$N(H) = \left\{x \in G \mid xHx^{-1} = H\right\}.$$

(Note that $xHx^{-1} = H$ means that the set $xHx^{-1}$ is equal to the set $H$.)

Prove that $N(H)$ is a subgroup of $G$.

**11.** (16 points) **PROOF QUESTION.** Let $R$ be a commutative ring, let $a$ be a fixed element of $R$, and let

$$A = \{x \in R \mid ax = 0\}.$$

Prove that $A$ is an ideal of $R$.

**12.** (16 points) **PROOF QUESTION.** Let $G$ be a group of order $143 = 11 \cdot 13$. Prove that $G$ contains an element of order 13.

**13.** (16 points) **PROOF QUESTION.** Suppose $\varphi : G \to \overline{G}$ is a homomorphism with $|\ker \varphi| = 2$, and suppose $a \in G$ has order 6.

(a) Prove that $\varphi(a)^6 = \overline{e}$, where $\overline{e}$ is the identity element of $\overline{G}$.

(b) Prove that the order of $\varphi(a)$ is either 6 or 3.