Sample Final Exam
Math 128A, Fall 2010

1. (13 points) Let R be a ring (not necessarily commutative).

(a) Define what it means for R to be a ring with unity.

(b) If we now assume that R is a ring with unity, define what it means for $a \in R$ to be a unit.

2. (17 points) Let G be a group whose operation is written multiplicatively, and let N be a normal subgroup of G. Define the quotient group (factor group) G/H. In particular, describe the **elements** of G/H, the **operation** in G/H, and the **identity element** of G/H.

3. (12 points) Let

$$\alpha = (2\ 5\ 3)(4\ 8\ 11\ 9\ 6)(7\ 10),$$
$$\beta = (1\ 10\ 3\ 4\ 8\ 11\ 2\ 7).$$

(a) Calculate $\alpha\beta$. Put your final answer in cycle form, and show all your work.

(b) Calculate $|\alpha\beta|$. Show all your work.

(c) Without further calculation, determine the order of $\beta^{-1}\alpha^{-1}$. Briefly **justify** your answer.

4. (12 points)

(a) **How many** abelian groups of order $360 = 2^3 \cdot 3^2 \cdot 5$ are there, up to isomorphism? Briefly justify your answer. (You do not need to list all such groups; just explain how you can be sure that your count is correct.)

(b) Give an example of an abelian group G of order 360 and an element $a \in G$ such that the order of a is 20. Briefly justify your answer.

For questions 5–10, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

5. (13 points) **(TRUE/FALSE)** Let R be an integral domain. Then it must be the case that every nonzero element of R is a unit.

6. (13 points) **(TRUE/FALSE)** It is possible that there exists an abelian group G and a surjective (onto) homomorphism $\varphi : G \to S_4$.

7. (13 points) **(TRUE/FALSE)** Let G be a group. If there exists an element $a \in G$ such that $\langle a \rangle = G$, and H is a subgroup of G, then it must be the case that H is cyclic.

8. (13 points) **(TRUE/FALSE)** Let G be a group of order 75. It is possible that there exists a subgroup H of G such that H is isomorphic to $U(10)$.

9. (13 points) **(TRUE/FALSE)** Let G be a group of order 12 that is not cyclic. Then it must be the case that G is isomorphic to D_6.

10. (13 points) **(TRUE/FALSE)** Let R be a commutative ring with unity. Then it must be the case that there are exactly two elements $a \in R$ such that $a^2 = 2a$.

11. (17 points) **PROOF QUESTION.** Let G be an abelian group whose operation is written multiplicatively. Note that the two parts of this question are connected.

(a) Let $\varphi : G \to G$ be defined by $\varphi(g) = g^4$. Prove that φ is a homomorphism.

(b) Let K be the set of all elements of G whose order divides 4, or in other words, let

$$K = \{g \in G \mid |g| \text{ divides } 4\}.$$

You may take it as given (i.e., do not try to prove) that K is a subgroup of G. Prove that there exists a subgroup H of G such that $G/K \approx H$ (i.e., prove that G/K is isomorphic to a subgroup of G).

12. (17 points) **PROOF QUESTION.** Let R be a commutative ring such that for all $r \in R$, $r + r = 0$, and let A be an ideal in R. Prove that for all $\overline{x}, \overline{y} \in R/A$, we have that $(\overline{x} + \overline{y})^2 = \overline{x}^2 + \overline{y}^2$.

13. (17 points) **PROOF QUESTION.** Recall that $GL(2, \mathbf{R})$ is the group of all invertible 2×2 matrices with real entries.

Let X be an arbitrary but fixed 2×2 matrix with real entries, and define

$$H(X) = \{A \in GL(2, \mathbf{R}) \mid AX = X\}.$$

Prove that $H(X)$ is a subgroup of $GL(2, \mathbf{R})$.

14. (17 points) **PROOF QUESTION.** Let G be a group of order $91 = 7 \cdot 13$. Prove that G contains at least one element of order 7.