Math 128A, problem set 09
CORRECTED THU NOV 04
Outline due: Wed Nov 10
Due: Mon Nov 15
Last revision due: Wed Dec 08

Problems to be done, but not turned in: (Ch. 10) 1, 3, 7, 9, 11, 17, 23, 27, 31, 39, 43, 47, 57.

Fun: (Ch. 10) 40, 52.

Problems to be turned in:

1. Recall (PS02, PS08, class) that if $r = R_{360/14}$ and f is any reflection in D_{14}, we know that $r^n = f^2 = e$ and $frf^{-1} = r^{-1}$, and that
$$D_{14} = \left\{ r^i, r^i f \mid 0 \le i < 14 \right\}.$$
Let $H = \langle r^7 \rangle = \left\{ e, r^7 \right\}$, and let $K = \left\{ r^{2i}, r^{2i} f \mid 0 \le i < 7 \right\}$.
   (a) Prove that H and K are normal in D_{14}.
   (b) Prove that $D_{14} \approx H \oplus K$.

   (While you do not need to prove this, it is a fact that $H \approx Z_2$ and $K \approx D_7$, which means that $D_{14} \approx Z_2 \oplus D_7$.)

2. Suppose we define a function $\varphi : \mathbf{Z}_6 \to \mathbf{Z}_{14}$ by the formula $\varphi(x) = 2x$, i.e., $\varphi(0) = 0$, $\varphi(1) = 2$, $\varphi(2) = 4$, $\varphi(3) = 6$, $\varphi(4) = 8$, $\varphi(5) = 10$. Is φ a homomorphism? Prove or disprove.

3. (Ch. 10) 18.

4. (Ch. 10) 22. You do not need to prove your generalization.

5. (Ch. 10) 32.

6. Let p be a prime, and let G be a group of order p^n for some $n \in \mathbf{Z}$, $n > 0$.
   (a) Think of $\text{Inn}(G)$ (see Ch. 6) as a group of permutations of the elements of G (i.e., as a subgroup of $\text{Sym}(G)$). In these terms, what is the orbit of e under $\text{Inn}(G)$ (see Ch. 7)? Prove your answer.
   (b) Prove that the order of $\text{Inn}(G)$ is p^k for some $k \in \mathbf{Z}$.
   (c) Prove that there exists some $a \in G$ such that the orbit of a under $\text{Inn}(G)$ is precisely $\{a\}$. (Suggestion: Orbit-Stabilizer.)
   (d) Prove that if G is a group of order p^n for some $n \in \mathbf{Z}$, $n > 0$, then there exists some nontrivial $a \in Z(G)$ (the center of G, see Ch. 3).

7. Suppose $\varphi : G \to D_6$ is a surjective homomorphism and $|\ker \varphi| = 13$. Prove that G has a normal subgroup of order 26 and a non-normal subgroup of order 26.