Math 128A, problem set 07
Outline due: Wed Oct 27
Due: Mon Nov 01
Last revision due: Mon Nov 15

Problems to be done, but not turned in: (Ch. 7) 19, 25, 27, 31, 33, 35, 41, 43, 47; (Ch. 8) 1, 5, 11, 15.

Fun: (Ch. 7) 32, 36.

Problems to be turned in:

1. (Ch. 7) 34.
2. (Ch. 7) 38.
3. Let G be a group, and let H and K be subgroups of G. Recall that
$$HK = \{hk \mid h \in H, k \in K\}.$$
Recall also that HK is sometimes a subgroup of G, and sometimes not.
   (a) Prove that for $a \in G$, the coset aK is partitioned into left cosets of $H \cap K$.
   (b) For $h_1, h_2 \in H$, find a condition involving h_1, h_2, and $H \cap K$ such that $h_1K = h_2K$ if and only if (condition). Prove your answer.
   (c) Prove that $|HK| = \dfrac{|H|\,|K|}{|H \cap K|}$.
4. Consider D_6 as a group of permutations of the vertices of a regular hexagon. For $1 \leq i \leq 6$, let
$$C_i = \{\alpha \in D_6 \mid \alpha(1) = i\}.$$
In other words, C_i is the set of all elements that send vertex 1 to vertex i.
For $1 \leq i \leq 6$, list all elements in C_i. What pattern(s) do you see? Do the sets C_i look familiar to you?
5. Let G be a group of permutations of $\{1, \ldots, n\}$, i.e., let G be a subgroup of S_n. Let $H = \text{stab}_G(1)$ and $K = \text{stab}_H(2)$, and suppose G, H, and K have the following properties:
   - $\text{orb}_G(1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.
   - $\text{orb}_H(2) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.
   - $\text{orb}_K(3) = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.
   - $\text{stab}_K(3) = \{e\}$.

   What is the order of G? Prove your answer.
6. Is $\mathbf{Z}_3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_9$ isomorphic to $\mathbf{Z}_9 \oplus \mathbf{Z}_9$? Why or why not? Prove your answer.
7. (Ch. 8) 22. Justify your answer. Your formula should be a numerical formula in terms of m and n.