Math 128A, problem set 05
CORRECTED MON OCT 05
Outline due: Wed Sep 30
Due: WED OCT 07
Last revision due: Mon Nov 23

Problems to be done, but not turned in: (Ch. 5) 21–67 odd; (Ch. 6) 1, 3, 5.

Problems to be turned in:

1. (Ch. 5) 26.

2. Let
$$H = \{\alpha \in S_8 \mid \alpha(i) \in \{1,2,3\} \text{ for all } i \in \{1,2,3\}\}.$$
Prove that H is a subgroup of S_8. (You may want to use the Pigeonhole Principle, in the following form: If X is a finite set, then $f : X \to X$ is one-to-one if and only if it is onto.)

3. This problem proves an analogue of Theorem 5.4.
   (a) Find two 3-cycles $\alpha, \beta \in S_4$ such that $\alpha\beta = (1\ 2)(3\ 4)$.
   (b) Prove that if β is a cycle of length $k \geq 3$, then there exists a 3-cycle α such that $\alpha\beta$ is a cycle of length $k-2$. (To make the notation less awkward, just do this for the k-cycle $(1\ 2\ \cdots\ k)$.)
   (c) Prove that if β is a cycle of length $k \geq 3$, then there exist 3-cycles $\alpha_1, \dots, \alpha_r$ such that $\alpha_r \cdots \alpha_1\beta$ is either the identity or a 2-cycle.
   (d) We have the following theorem:

   Theorem (*): If β is an even permutation, then then there exist 3-cycles $\alpha_1, \dots, \alpha_s$ such that $\alpha_s \cdots \alpha_1\beta = \epsilon$.

   To make the notation less complicated, prove Theorem (*) in the case where
   $$\beta = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9)(10\ 11\ 12\ 13\ 14\ 15\ 16)(17\ 18),$$
   and explain how you know your method works in general.
   (e) Now suppose Theorem (*) holds in general, and not just in the special case you proved. Prove that every even permutation is a product of 3-cycles (not necessarily disjoint).

4. *Revised version:* Let H be a subgroup of S_n that contains at least one odd permutation $\alpha \in H$, let $K = A_n \cap H$ be the subgroup of H that contains the even permutations in H, and let $\mathcal{O}$ be the set of all odd permutations of H.
   (a) Prove that if $x \in K$, then $\alpha x \in \mathcal{O}$.
   (b) Now define a map $f : K \to \mathcal{O}$ by the formula $f(x) = \alpha x$. Prove that f is a bijection.
   (c) Prove that $|H| = 2\,|K|$. (It follows that the order of H is even.)

Old version:

(a) Let G be a group, and let a be an element of G. Prove that the map $f : G \to G$ defined by $f(x) = ax$ is a bijection (one-to-one and onto).

(b) Let H be a subgroup of S_n that contains at least one odd permutation. Prove that the order of H is even.

5. (Ch. 6) 6.

6. (a) Prove that the map $\varphi_3 : U(11) \to U(11)$ defined by $\varphi_3(x) = x^3$ is an automorphism of $U(11)$.

   (b) For which n is the map $\varphi_n : U(11) \to U(11)$ defined by $\varphi_n(x) = x^n$ an automorphism of $U(11)$? Find a pattern. (You do not need to prove your answer to 6b, but show some work or evidence.)