Math 128A, Wed Oct 21

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today and Monday: Ch. 9.
- PS07 outline due Monday, full version due in 1 week.
- Problem session, Fri Oct 23, 10:00–noon on Zoom.

Two fundamental questions

$\varphi(ab) = \varphi(a)\varphi(b)$

Chs. 9 and 10 are about:

- (Ch. 9) Given a group G and a subgroup H of G, can we turn the (left) cosets aH of H in G into a group, using the operation of G?
- (Ch. 10) Given an operation-preserving map, or **homomorphism**, $\varphi : G \to H$, that is **not** necessarily a bijection, what can we say about how the structure of G relates to the structure of H?

Second question is more natural than the first; but it turns out that the answer to the second question is best expressed in terms of the first, so we'll start there.

Example

The index of H in G $= \dfrac{|G|}{|H|}$

Let $G = D_6$, and let

$$H = \langle R_{60} \rangle = \{e, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}\}.$$

$|G : H| = 2$, so there are two cosets of H in G: H itself, and

$$F_1 H = \{F_1, F_2, F_3, F_{12}, F_{23}, F_{34}\} = F_2 H = \cdots$$

Q: If we multiply one coset times another (a la HK), is the set of all elements we get a coset? (of H)

Ex

$$(H)(F_1H) = \left\{ \begin{matrix} F_1, F_2, F_3 \\ F_{12}, F_{23}, F_{34} \end{matrix} \right\} = F_1 H$$

H FH

	D_6	e	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}	F_1	F_2	F_3	F_{12}	F_{23}	F_{34}
	e	e	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}	F_1	F_2	F_3	F_{12}	F_{23}	F_{34}
	R_{60}	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}	e	F_{12}	F_{23}	F_{34}	F_2	F_3	F_1
H	R_{120}	R_{120}	R_{180}	R_{240}	R_{300}	e	R_{60}	F_2	F_3	F_1	F_{23}	F_{34}	F_{12}
	R_{180}	R_{180}	R_{240}	R_{300}	e	R_{60}	R_{120}	F_{23}	F_{34}	F_{12}	F_3	F_1	F_2
	R_{240}	R_{240}	R_{300}	e	R_{60}	R_{120}	R_{180}	F_3	F_1	F_2	F_{34}	F_{12}	F_{23}
	R_{300}	R_{300}	e	R_{60}	R_{120}	R_{180}	R_{240}	F_{34}	F_{12}	F_{23}	F_1	F_2	F_3
	F_1	F_1	F_{34}	F_3	F_{23}	F_2	F_{12}	e	R_{240}	R_{120}	R_{300}	R_{180}	R_{60}
	F_2	F_2	F_{12}	F_1	F_{34}	F_3	F_{23}	R_{120}	e	R_{240}	R_{60}	R_{300}	R_{180}
	F_3	F_3	F_{23}	F_2	F_{12}	F_1	F_{34}	R_{240}	R_{120}	e	R_{180}	R_{60}	R_{300}
FH	F_{12}	F_{12}	F_1	F_{34}	F_3	F_{23}	F_2	R_{60}	R_{300}	R_{180}	e	R_{240}	R_{120}
	F_{23}	F_{23}	F_2	F_{12}	F_1	F_{34}	F_3	R_{180}	R_{60}	R_{300}	R_{120}	e	R_{240}
	F_{34}	F_{34}	F_3	F_{23}	F_2	F_{12}	F_1	R_{300}	R_{180}	R_{60}	R_{240}	R_{120}	e

G/H	H	F_1H
H	H	F_1H
F_1H	F_1H	H

$$\boxed{G/H \approx \mathbb{Z}_2}$$

$|a|=2$

	e	a
e	e	a
a	a	$a^2=e$

$e=H$

$\mathbb{Z}_2$	0	1
0	0	1
1	1	0

	e	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}	F_1	F_2	F_3	F_{12}	F_{23}	F_{34}
e	e	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}	F_1	F_2	F_3	F_{12}	F_{23}	F_{34}
R_{60}	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}	e	F_{12}	F_{23}	F_{34}	F_2	F_3	F_1
R_{120}	R_{120}	R_{180}	R_{240}	R_{300}	e	R_{60}	F_2	F_3	F_1	F_{23}	F_{34}	F_{12}
R_{180}	R_{180}	R_{240}	R_{300}	e	R_{60}	R_{120}	F_{23}	F_{34}	F_{12}	F_3	F_1	F_2
R_{240}	R_{240}	R_{300}	e	R_{60}	R_{120}	R_{180}	F_3	F_1	F_2	F_{34}	F_{12}	F_{23}
R_{300}	R_{300}	e	R_{60}	R_{120}	R_{180}	R_{240}	F_{34}	F_{12}	F_{23}	F_1	F_2	F_3
F_1	F_1	F_{34}	F_3	F_{23}	F_2	F_{12}	e	R_{240}	R_{120}	R_{300}	R_{180}	R_{60}
F_2	F_2	F_{12}	F_1	F_{34}	F_3	F_{23}	R_{120}	e	R_{240}	R_{60}	R_{300}	R_{180}
F_3	F_3	F_{23}	F_2	F_{12}	F_1	F_{34}	R_{240}	R_{120}	e	R_{180}	R_{60}	R_{300}
F_{12}	F_{12}	F_1	F_{34}	F_3	F_{23}	F_2	R_{60}	R_{300}	R_{180}	e	R_{240}	R_{120}
F_{23}	F_{23}	F_2	F_{12}	F_1	F_{34}	F_3	R_{180}	R_{60}	R_{300}	R_{120}	e	R_{240}
F_{34}	F_{34}	F_3	F_{23}	F_2	F_{12}	F_1	R_{300}	R_{180}	R_{60}	R_{240}	R_{120}	e

Two more examples

$G = D_6$, and now consider

$$H_1 = \{e, R_{180}, F_2, F_{34}\},$$
$$H_2 = \{e, R_{180}\}. = \langle R_{180} \rangle$$

Next two slides show Cayley table of G rewritten in terms of cosets of H_1 and H_2. Again we ask:

- If you multiply two cosets of H_1 together, do you get a coset of H_1? I.e., is $aH_1 bH_1 = cH_1$ for some $c \in G$?
- Same, but for H_2.

H_1 F_1H_1 F_3H_1

D_6	e	R_{180}	F_2	F_{34}	R_{60}	R_{240}	F_1	F_{23}	R_{120}	R_{300}	F_3	F_{12}
e	e	R_{180}	F_2	F_{34}	R_{60}	R_{240}	F_1	F_{23}	R_{120}	R_{300}	F_3	F_{12}
R_{180}	R_{180}	e	F_{34}	F_2	R_{240}	R_{60}	F_{23}	F_1	R_{300}	R_{120}	F_{12}	F_3
F_2	F_2	F_{34}	e	R_{180}	F_{12}	F_3	R_{120}	R_{300}	F_1	F_{23}	R_{240}	R_{60}
F_{34}	F_{34}	F_2	R_{180}	e	F_3	F_{12}	R_{300}	R_{120}	F_{23}	F_1	R_{60}	R_{240}
R_{60}	R_{60}	R_{240}	F_{23}	F_1	R_{120}	R_{300}	F_{12}	F_3	R_{180}	e	F_{34}	F_2
F_1H R_{240}	R_{240}	R_{60}	F_1	F_{23}	R_{300}	R_{120}	F_3	F_{12}	e	R_{180}	F_2	F_{34}
F_1	F_1	F_{23}	R_{240}	R_{60}	F_{34}	F_2	e	R_{180}	F_3	F_{12}	R_{120}	R_{300}
F_{23}	F_{23}	F_1	R_{60}	R_{240}	F_2	F_{34}	R_{180}	e	F_{12}	F_3	R_{300}	R_{120}
R_{120}	R_{120}	R_{300}	F_3	F_{12}	R_{180}	e	F_2	F_{34}	R_{240}	R_{60}	F_1	F_{23}
R_{300}	R_{300}	R_{120}	F_{12}	F_3	e	R_{180}	F_{34}	F_2	R_{60}	R_{240}	F_{23}	F_1
F_3	F_3	F_{12}	R_{120}	R_{300}	F_{23}	F_1	R_{240}	R_{60}	F_2	F_{34}	e	R_{180}
F_{12}	F_{12}	F_3	R_{300}	R_{120}	F_1	F_{23}	R_{60}	R_{240}	F_{34}	F_2	R_{180}	e

H_2 $F_2 H_2$

	e	R_{180}	F_2	F_{34}	R_{60}	R_{240}	F_1	F_{23}	R_{120}	R_{300}	F_3	F_{12}
e	e	R_{180}	F_2	F_{34}	R_{60}	R_{240}	F_1	F_{23}	R_{120}	R_{300}	F_3	F_{12}
R_{180}	R_{180}	e	F_{34}	F_2	R_{240}	R_{60}	F_{23}	F_1	R_{300}	R_{120}	F_{12}	F_3
F_2	F_2	F_{34}	e	R_{180}	F_{12}	F_3	R_{120}	R_{300}	F_1	F_{23}	R_{240}	R_{60}
F_{34}	F_{34}	F_2	R_{180}	e	F_3	F_{12}	R_{300}	R_{120}	F_{23}	F_1	R_{60}	R_{240}
R_{60}	R_{60}	R_{240}	F_{23}	F_1	R_{120}	R_{300}	F_{12}	F_3	R_{180}	e	F_{34}	F_2
R_{240}	R_{240}	R_{60}	F_1	F_{23}	R_{300}	R_{120}	F_3	F_{12}	e	R_{180}	F_2	F_{34}
F_1	F_1	F_{23}	R_{240}	R_{60}	F_{34}	F_2	e	R_{180}	F_3	F_{12}	R_{120}	R_{300}
F_{23}	F_{23}	F_1	R_{60}	R_{240}	F_2	F_{34}	R_{180}	e	F_{12}	F_3	R_{300}	R_{120}
R_{120}	R_{120}	R_{300}	F_3	F_{12}	R_{180}	e	F_2	F_{34}	R_{240}	R_{60}	F_1	F_{23}
R_{300}	R_{300}	R_{120}	F_{12}	F_3	e	R_{180}	F_{34}	F_2	R_{60}	R_{240}	F_{23}	F_1
F_3	F_3	F_{12}	R_{120}	R_{300}	F_{23}	F_1	R_{240}	R_{60}	F_2	F_{34}	e	R_{180}
F_{12}	F_{12}	F_3	R_{300}	R_{120}	F_1	F_{23}	R_{60}	R_{240}	F_{34}	F_2	R_{180}	e

The answers

$G = D_6$,

$$H_1 = \{e, R_{180}, F_2, F_{34}\},$$
$$H_2 = \{e, R_{180}\}.$$

- The product of two cosets of H_1 need not be a coset of H_1.
- But the product of two cosets of H_2 is always a coset of H_2.

Q: Given $H \leq G$, how can we determine if the product of two (left) cosets of H is always a (left) coset of H?
A: This happens exactly when H is **normal**.

$R_{120}H_1$

	e	R_{180}	F_2	F_{34}	R_{60}	R_{240}	F_1	F_{23}	R_{120}	R_{300}	F_3	F_{12}
e	e	R_{180}	F_2	F_{34}	R_{60}	R_{240}	F_1	F_{23}	R_{120}	R_{300}	F_3	F_{12}
R_{180}	R_{180}	e	F_{34}	F_2	R_{240}	R_{60}	F_{23}	F_1	R_{300}	R_{120}	F_{12}	F_3
F_2	F_2	F_{34}	e	R_{180}	F_{12}	F_3	R_{120}	R_{300}	F_1	F_{23}	R_{240}	R_{60}
F_{34}	F_{34}	F_2	R_{180}	e	F_3	F_{12}	R_{300}	R_{120}	F_{23}	F_1	R_{60}	R_{240}
R_{60}	R_{60}	R_{240}	F_{23}	F_1	R_{120}	R_{300}	F_{12}	F_3	R_{180}	e	F_{34}	F_2
R_{240}	R_{240}	R_{60}	F_1	F_{23}	R_{300}	R_{120}	F_3	F_{12}	e	R_{180}	F_2	F_{34}
F_1	F_1	F_{23}	R_{240}	R_{60}	F_{34}	F_2	e	R_{180}	F_3	F_{12}	R_{120}	R_{300}
F_{23}	F_{23}	F_1	R_{60}	R_{240}	F_2	F_{34}	R_{180}	e	F_{12}	F_3	R_{300}	R_{120}
R_{120}	R_{120}	R_{300}	F_3	F_{12}	R_{180}	e	F_2	F_{34}	R_{240}	R_{60}	F_1	F_{23}
R_{300}	R_{300}	R_{120}	F_{12}	F_3	e	R_{180}	F_{34}	F_2	R_{60}	R_{240}	F_{23}	F_1
F_3	F_3	F_{12}	R_{120}	R_{300}	F_{23}	F_1	R_{240}	R_{60}	F_2	F_{34}	e	R_{180}
F_{12}	F_{12}	F_3	R_{300}	R_{120}	F_1	F_{23}	R_{60}	R_{240}	F_{34}	F_2	R_{180}	e

F_1H_1

H_2

	e	R_{180}	F_2	F_{34}	R_{60}	R_{240}	F_1	F_{23}	R_{120}	R_{300}	F_3	F_{12}
e	e	R_{180}	F_2	F_{34}	R_{60}	R_{240}	F_1	F_{23}	R_{120}	R_{300}	F_3	F_{12}
R_{180}	R_{180}	e	F_{34}	F_2	R_{240}	R_{60}	F_{23}	F_1	R_{300}	R_{120}	F_{12}	F_3
F_2	F_2	F_{34}	e	R_{180}	F_{12}	F_3	R_{120}	R_{300}	F_1	F_{23}	R_{240}	R_{60}
F_{34}	F_{34}	F_2	R_{180}	e	F_3	F_{12}	R_{300}	R_{120}	F_{23}	F_1	R_{60}	R_{240}
R_{60}	R_{60}	R_{240}	F_{23}	F_1	R_{120}	R_{300}	F_{12}	F_3	R_{180}	e	F_{34}	F_2
R_{240}	R_{240}	R_{60}	F_1	F_{23}	R_{300}	R_{120}	F_3	F_{12}	e	R_{180}	F_2	F_{34}
F_1	F_1	F_{23}	R_{240}	R_{60}	F_{34}	F_2	e	R_{180}	F_3	F_{12}	R_{120}	R_{300}
F_{23}	F_{23}	F_1	R_{60}	R_{240}	F_2	F_{34}	R_{180}	e	F_{12}	F_3	R_{300}	R_{120}
R_{120}	R_{120}	R_{300}	F_3	F_{12}	R_{180}	e	F_2	F_{34}	R_{240}	R_{60}	F_1	F_{23}
R_{300}	R_{300}	R_{120}	F_{12}	F_3	e	R_{180}	F_{34}	F_2	R_{60}	R_{240}	F_{23}	F_1
F_3	F_3	F_{12}	R_{120}	R_{300}	F_{23}	F_1	R_{240}	R_{60}	F_2	F_{34}	e	R_{180}
F_{12}	F_{12}	F_3	R_{300}	R_{120}	F_1	F_{23}	R_{60}	R_{240}	F_{34}	F_2	R_{180}	e

Normal subgroups

I.e., left and right cosets are always the same.

Definition

To say that $H \leq G$ is **normal** in G means that $aH = Ha$ for all $a \in G$, in which case we write $H \lhd G$.

"H is a normal subgroup of G"

Examples: $G = D_6$.

- For $H = \langle R_{90} \rangle$ and $H = \langle R_{180} \rangle$, we have $aH = Ha$ for all $a \in G$. $= \{e, R_{90}, \ldots\}$ $\{e, R_{180}\}$
- But for $H = \{e, R_{180}, F_2, F_{34}\}$, we have

(see H_1)

$$F_1 H = \{F_1, F_{27}, R_{240}, R_{60}\} \neq$$

Not normal

$$HF_1 = \{F_1, F_{23}, R_{120}, R_{300}\}$$

Observation: If G is Abelian, then $aH = Ha$, and so all subgroups are normal.

This appears most commonly if operation in G is $+$, in which case we see that $a + H = H + a$.

In particular, all subgroups of cyclic groups are normal.

Normal subgroup test

Theorem

Suppose $H \leq G$. TFAE:

1. $H \lhd G$.
2. *For all* $x \in H$, $x^{-1}Hx \subseteq H$.

~~**Proof.** ($1 \Rightarrow 2$)~~

$h \to x^{-1}hx$

<u>conjugation</u>

Example: Let $G = D_6$, $H = \langle R_{60} \rangle$. Prove $H \lhd G$.

Try all x:

$x = R_d$: Rot'ns comm

$H = \{e, R_{60}, \ldots, R_{240}\}$

$R_d^{-1} R_{60} R_d = R_d^{-1} R_d R_{60} = R_{60}$

$R_d^{-1} H R_d = H$, in same order

If $x = F$ refl, $F^{-1} R_d F = R_d^{-1}$

$F^{-1} H F = \{e, R_{300}, R_{240}, \ldots, R_{60}\}$

$= H$ (in opp order)

Factor groups

Definition

For $H \lhd G$, the **factor group**, or **quotient group**, G/H is:

- **Set:** All (left) cosets aH. (Same as right cosets Ha because $aH = Ha$.)
- **Operation:** We define

$$(aH)(bH) = (ab)H.$$

Note that this is the multiplication of cosets that you get when you multiply individual elements — assuming that coset times coset is coset.

Theorem

G/H really is a group.

Proof: Hard part is showing that operation is well-defined; i.e., if $aH = a'H$ and $bH = b'H$, is $(a'b')H = (ab)H$?

Example

$G = D_6$, $H = \langle R_{60} \rangle$. Then

$$G/H =$$

$G = D_6$, $H = \langle R_{180} \rangle$. Then

$$G/H =$$

G/Z theorem

Theorem

G a group, $Z = Z(G)$ center of G. If G/Z is cyclic, then G is abelian.

Proof: Suppose G/Z is cyclic. Then G/Z is generated by some coset aZ, i.e.: