Math 128A, Wed Sep 23

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: Ch. 5. Reading for Mon: Ch. 6.
- Outline for PS04 due tonight; completed version due Mon Sep 28.
- Problem session Fri Sep 25, 10:00–noon on Zoom.

Cycle notation for permutations

Theorem

Every permutation is a product of disjoint cycles.

Proof by (an example of) algorithm: In S_{12}, take

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 4 & 7 & 3 & 6 & 8 & 11 & 2 & 12 & 5 & 1 & 10 & 9 \end{pmatrix}$$

Then starting with 1:

- Take the smallest number i not yet included in a cycle;
- Figure out the cycle containing i; and
- Repeat until every element of $\{1, \ldots, 12\}$ is in a cycle.

often omit fixed pts

$$\alpha = (1\ 4\ 6\ 11\ 10)(2\ 7)(3)(5\ 8\ 12\ 9)$$

cycle form of alpha

$$\alpha = (1\ 4\ 6\ 11\ 10)(2\ 7)(5\ 8\ 12\ 9)$$

You try

Convention: Each cycle is written with earliest number first, and cycles sorted by their starting number. This makes each permutation have a unique cycle form.

$(1\,2\,3) = (2\,3\,1) = (3\,1\,2)$

Definition

The **cycle form** of a permutation α is α expressed as a product of disjoint cycles.

Given:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 4 & 2 & 6 & 7 & 1 & 3 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 5 & 2 & 8 & 7 \end{pmatrix}$$

Compute $\alpha\beta$ and the cycle forms of α, β, and $\alpha\beta$.

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 2 & 8 & 6 & 5 & 3 & 1 \end{pmatrix}$$

Next-level computation: Computing products in cycle form

$$\alpha = (1\ 8\ 3\ 4\ 2\ 5\ 6\ 7)$$

$$\beta = (1\ 3\ 4)(2\ 6)(7\ 8)$$

$$\alpha\beta = (1_3, 4_1, 8_7)(2_6, 7_8, 3_4)(5_5, 6_2)$$

If you want to stick with converting to the 2-line form, that's fine!
But if you want to get faster, this is the way to do it.

Permutations in cycle form

Definition

The **cycle form** of a permutation α is α expressed as a product of disjoint cycles.

Theorem

Disjoint cycles commute.

If α, β disj cycles, $\alpha\beta = \beta\alpha$

Proof by playing cards!

Order of a permutation in cycle form

Theorem

The order of a permutation written in cycle form is the LCM of its cycle lengths.

Proof: Suppose $\alpha = \beta\gamma$, where β and γ are disjoint cycles. Because disjoint cycles commute:

$$\alpha^n = \beta^n \gamma^n.$$

β and γ permute disjoint sets, so to get $\alpha^n = \epsilon$, need to have $\beta^n = \gamma^n = \epsilon$. So n must be a common multiple of the lengths of β and γ, and the smallest such n is the least common multiple of the cycle lengths.

pf by cavAs

Examples

3 2 2

Orders of elements: $\alpha = (1\ 8\ 3\ 4\ 2\ 5\ 6\ 7)$, $\beta = (1\ 3\ 4)(2\ 6)(7\ 8)$.

$\mathrm{ord}(\alpha) = 8$

$\mathrm{ord}(\beta) = \mathrm{LCM}(3,2,2) = 6$

Possible cycle shapes of elements of S_5, and their orders: order

$(a\ b\ c\ d\ e)$ 5 5

$(a\ b\ c\ d)(e)$ 4+1 4

$(a\ b\ c)(d\ e)$ 3+2 6

$(a\ b\ c)(d)(e)$ 3+1+1 3

Ordering ways to sum positive integers, sorted in decreasing order, to a total of 5
(# of ways to do this is called # of *partitions* of 5.)

$(a\,b)(c\,d)(e)$ $\quad 2+2+1 \quad\quad 2$

$(a\,b)(c)(d)(e)$ $\quad 2+1+1+1 \quad\quad 2$

$(a)(b)(c)(d)(e)$ $\quad 1+1+1+1+1 \quad\quad 1$

p(n) = # of partitions of n, in the above sense.

If you could write down an efficient formula for computing p(n), you could get a job for life at a university or the NSA (or organized crime).

Products of 2-cycles

Theorem

Every $\alpha \in S_n$ is a product of 2-cycles.

Proof: Consider

$$\boxed{(1\ 2)(2\ 3)(3\ 4)(4\ 5) = (1\,2\,3\,4\,5)}$$

Same pattern shows that any k-cycle is the product of $k-1$ 2-cycles.

And then recall that any permutation is the product of k-cycles

Even and odd permutations

Lemma

If $\epsilon = \beta_1\beta_2\ldots\beta_r$*, where each* β_i *is a 2-cycle, then* r *is even.*

Theorem

For $\alpha \in S_n$*, exactly one of the following is true:*

- α *is a product of an* **even** *number of 2-cycles; or*
- α *is a product of an* **odd** *number of 2-cycles.*

Proof: Suppose

$$\alpha = \beta_1\ldots\beta_k = \gamma_1\ldots\gamma_m,$$

where each β_i and γ_j is a 2-cycle.

The alternating group

Definition

If α is product of an even number of 2-cycles, we say α is **even**; if α is product of an odd number of 2-cycles, we say α is **odd**.

Prev thm says that a permutation is either odd or even, but not both.
Fact (Thm)/Defn: Even permutations in S_n form a subgroup of S_n called the **alternating group** of degree n, written A_n.
Why subgroup:

Size of A_n

Theorem

For $n \geq 2$, A_n is exactly half the size of S_n, i.e., $|A_n| = \dfrac{n!}{2}$.

Proof: Consider the set

$$(1\ 2)A_n =$$

Cycles as odd and even permutations

- Cycles of odd length are
- Cycles of even length are
- So if α is a product of disjoint cycles, α is an even permutation $\Leftrightarrow$