Sample Exam 2
Math 127, Fall 2024

1. (12 points) Let W be a subspace of $\mathbf{F}_7^{13}$ such that $\dim W = 5$.

(a) What is the number of vectors in a smallest possible spanning set for W? Briefly **EXPLAIN** your answer.

(b) Is it possible that there exists a linearly independent subset $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ of W? Briefly **EXPLAIN** why or why not.

2. (12 points) Let

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \\ 0 \end{bmatrix} \middle| \; x_1, x_2, x_3 \in \mathbf{F}_{19} \right\}.$$

You may take it as given that W is a subspace of $\mathbf{F}_{19}^5$.

(a) Guess a basis $\mathcal{B}$ for W. No explanation necessary.

(b) Use the **definition** of linear independence to prove that $\mathcal{B}$ is linearly independent.

3. (12 points) Let $\mathcal{C}$ be the binary linear code of length 7 with parity check matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

(a) Find a generator matrix for $\mathcal{C}$. Show your work.

(b) What is the dimension of $\mathcal{C}$? Briefly **EXPLAIN** your answer.

4. (12 points) Consider $a(x) = x^3 + x^2$ and $b(x) = x^2 + x + 1$ in $\mathbf{F}_2[x]$, and note that

$$\begin{aligned} x^3 + x^2 &= (x)(x^2 + x + 1) + x, \\ x^2 + x + 1 &= (x + 1)(x) + 1, \\ x &= (x)(1). \end{aligned}$$

(I.e., you are given the above facts and do not need to check them yourself.)

Find $f(x), g(x) \in \mathbf{F}_2[x]$ such that $f(x)a(x) + g(x)b(x) = \gcd(a(x), b(x))$. Show your work and clearly indicate your answer.

5. (13 points) Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 2 \\ 5 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 4 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

in $\mathbf{F}_7^6$ and let $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(a) What is the dimension of W? Briefly **JUSTIFY** your answer.

(b) How many vectors are in W? Briefly **EXPLAIN** your answer.

6. (13 points) Let A be a matrix with entries in $\mathbf{F}_5$ such that

$$A = \begin{bmatrix} 0 & 2 & 2 & 4 & 2 & 1 & 2 & 3 \\ 2 & 2 & 3 & 2 & 3 & 1 & 0 & 2 \\ 4 & 4 & 1 & 0 & 0 & 3 & 0 & 1 \\ 2 & 2 & 3 & 3 & 2 & 4 & 3 & 3 \\ 2 & 0 & 1 & 0 & 4 & 0 & 2 & 3 \\ 3 & 4 & 3 & 1 & 2 & 4 & 0 & 1 \end{bmatrix}, \qquad RREF(A) = \begin{bmatrix} 1 & 0 & 3 & 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 4 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find bases for $\text{Col}(A)$ and $\text{Null}(A)$. Show your work.

7. (13 points) Recall that the parity check matrix of the Hamming 7-code $\mathcal{H}_7$ is

$$H_7 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Suppose Yolanda is receiving transmissions sent using the Hamming 7-code, and she receives $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Correct $\mathbf{y}$ to a codeword $\mathbf{y}'$, if necessary, and read off the message bits 3, 5, 6, and 7 to find the intended message $\mathbf{m}'$. Show all your work.

8. (13 points) Let W be the subset of $\mathbf{F}_{17}^3$ defined by

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{F}_{17}^3 \,\middle|\, x_1 + x_2 = 0 \right\}.$$

Give **part of** the proof that W is a subspace of $\mathbf{F}_{17}^3$, in the following steps:

(a) Explain why $\mathbf{0} \in W$.

(b) Suppose $\mathbf{x}, \mathbf{y} \in W$. Explain why $\mathbf{x} + \mathbf{y} \in W$.