Sample Exam 2
Math 127, Fall 2023

1. (12 points) Let $\mathcal{C}$ be a binary linear code of length n.

(a) Define what it means for a matrix G to be a generator matrix for $\mathcal{C}$.

(b) If G is a generator matrix for $\mathcal{C}$ and H is a parity check matrix for $\mathcal{C}$, what can you say about the product HG? Briefly **EXPLAIN** your answer.

2. (12 points) Let

$$W = \left\{ \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} \middle| \, x_1, x_3, x_4 \in \mathbf{F}_{23} \right\}.$$

You may take it as given that W is a subspace of $\mathbf{F}_{23}^4$.

(a) Guess a basis $\mathcal{B}$ for W. No explanation necessary.

(b) Prove that $\mathcal{B}$ spans W, i.e., prove that every $\mathbf{x} \in W$ is a linear combination of the vectors in $\mathcal{B}$.

3. (12 points) Consider $a(x) = x^6 + x^2$ and $b(x) = x^5 + x^4 + x^3 + 1$ in $\mathbf{F}_2[x]$, and note that

$$\begin{aligned} x^6 + x^2 &= (x+1)(x^5 + x^4 + x^3 + 1) + x^3 + x^2 + x + 1, \\ x^5 + x^4 + x^3 + 1 &= (x^2)(x^3 + x^2 + x + 1) + x^2 + 1, \\ x^3 + x^2 + x + 1 &= (x+1)(x^2+1). \end{aligned}$$

(I.e., you are given the above facts and do not need to check them yourself.)

Find $f(x), g(x) \in \mathbf{F}_2[x]$ such that $f(x)a(x) + g(x)b(x) = \gcd(a(x), b(x))$. Show your work and clearly indicate your answer.

4. (12 points) Let $\mathcal{C}$ be the binary linear code of length 7 with parity check matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

(a) Find a basis for $\mathcal{C}$. Show your work.

(b) What is the dimension of $\mathcal{C}$? Briefly **EXPLAIN** your answer.

5. (13 points) Recall that the parity check matrix of the Hamming 7-code $\mathcal{H}_7$ is

$$H_7 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Suppose Yolanda is receiving transmissions sent using the Hamming 7-code, and she receives $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. Correct $\mathbf{y}$ to a codeword $\mathbf{y}'$, if necessary, and read off the message bits 3, 5, 6, and 7 to find the intended message $\mathbf{m}'$. Show all your work.

6. (13 points) Let W be the subset of $\mathbf{F}_{11}^3$ defined by

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{F}_{11}^3 \,\middle|\, x_3 = 2x_1 \right\}.$$

Give **part of** the proof that W is a subspace of $\mathbf{F}_{11}^3$, in the following steps:

(a) Explain why $\mathbf{0} \in W$.

(b) Suppose $\mathbf{x} \in W$ and $a \in \mathbf{F}_{11}$. Explain why $a\mathbf{x} \in W$.

7. (13 points) Let A be a matrix with entries in $\mathbf{F}_7$ such that

$$A = \begin{bmatrix} 4 & 3 & 5 & 5 & 3 & 4 & 0 \\ 5 & 4 & 0 & 4 & 1 & 2 & 1 \\ 1 & 6 & 3 & 2 & 2 & 0 & 2 \\ 1 & 3 & 1 & 3 & 1 & 1 & 3 \\ 0 & 4 & 5 & 5 & 1 & 4 & 5 \end{bmatrix}, \qquad RREF(A) = \begin{bmatrix} 1 & 0 & 6 & 0 & 5 & 0 & 2 \\ 0 & 1 & 3 & 0 & 4 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find bases for $\mathrm{Col}(A)$ and $\mathrm{Null}(A)$. Show your work.

8. (13 points) Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

in $\mathbf{F}_5^4$ and let $W = \mathrm{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(a) What is the dimension of W? **PROVE** your answer using the **definition** of linear independence, and not row reduction.

(b) How many vectors are in W? Briefly **EXPLAIN** your answer.