Sample Exam 2
Math 127, Spring 2023

1. (12 points)

(a) Define what it means for $\mathcal{C}$ to be a binary linear code of length n.

(b) Let $\mathcal{C}$ be a binary linear code of length n. Define what it means for a matrix H to be a parity check matrix of $\mathcal{C}$.

2. (12 points) Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \end{bmatrix},$$

with entries in $\mathbf{F}_3$. Find the RREF of A. Show all your work.

3. (12 points) Let

$$W = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \end{bmatrix} \middle| \, x_1, x_4 \in \mathbf{F}_{17} \right\}.$$

You may take it as given that W is a subspace of $\mathbf{F}_{17}^4$.

(a) Guess a basis $\mathcal{B}$ for W. No explanation necessary.

(b) Prove that $\mathcal{B}$ spans W, i.e., prove that every $\mathbf{x} \in W$ is a linear combination of the vectors in $\mathcal{B}$.

4. (12 points) Consider $a(x) = x^5 + x^4 + x^2 + 1$ and $b(x) = x^4 + x^3$ in $\mathbf{F}_2[x]$, and note that

$$\begin{aligned} x^5 + x^4 + x^2 + 1 &= x(x^4 + x^3) + (x^2 + 1), \\ x^4 + x^3 &= (x^2 + x + 1)(x^2 + 1) + (x + 1), \\ x^2 + 1 &= (x + 1)(x + 1). \end{aligned}$$

(I.e., you are given the above facts and do not need to check them yourself.)

Find $f(x), g(x) \in \mathbf{F}_2[x]$ such that $f(x)a(x) + g(x)b(x) = \gcd(a(x), b(x))$. Show your work and clearly indicate your answer.

5. (13 points) Recall that the parity check matrix of the Hamming 7-code $\mathcal{H}_7$ is

$$H_7 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Suppose Yolanda is receiving transmissions sent using the Hamming 7-code, and she receives $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Correct $\mathbf{y}$ to a codeword $\mathbf{y}'$, if necessary, and read off the message bits 3, 5, 6, and 7 to find the intended message $\mathbf{m}'$. Show all your work.

6. (13 points) Let A be a matrix with entries in $\mathbf{F}_7$ such that

$$A = \begin{bmatrix} 3 & 1 & 2 & 0 & 0 \\ 1 & 5 & 2 & 5 & 2 \\ 2 & 3 & 0 & 4 & 4 \\ 1 & 5 & 0 & 0 & 3 \end{bmatrix}, \qquad RREF(A) = \begin{bmatrix} 1 & 5 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find bases for Col(A) and Null(A). Show your work.

7. (13 points) Let W be the subset of $\mathbf{F}_7^3$ defined by

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{F}_{19}^3 \;\middle|\; x_1 + 5x_2 = 0 \right\}.$$

Give **part of** the proof that W is a subspace of $\mathbf{F}_{19}^3$, in the following steps:

(a) Explain why $\mathbf{0} \in W$.

(b) Suppose $\mathbf{x}, \mathbf{y} \in W$. Explain why $\mathbf{x} + \mathbf{y} \in W$.

8. (13 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be three vectors in $\mathbf{F}_{11}^4$ and let $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(a) What is the largest possible number of vectors that W could contain? Briefly **EXPLAIN** your answer. (You can just express your answer as a power of some number; you don't have to multiply out that power.)

(b) What condition can you put on the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ that will ensure that W contains that largest possible number of vectors? Briefly **EXPLAIN** your answer.