Math 127, Mon Mar 08

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 5.5–5.6 (reload book again). 5.5: all new problems!

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- Reading for Wed: 6.1–6.2.
- PS04 due tonight; PS05 outline due Wed Mar 10.
- Problem session Fri Mar 12, 10am–noon.

Linear algebra: Questions to resolve F field

- Given a subspace W of Fⁿ and vectors {v₁,..., v_k} that span W, how can we check that {v₁,..., v_k} is a basis for W?
- Given a subspace W of F^n , how can we find a basis for W?
- Is it possible for a subspace W to have one basis with 5 vectors and another basis with 7 vectors? In other words, is it possible for the dimension of W to be both 5 and 7?
- Is it possible for F⁸ to contain a subspace of dimension 10? In other words, is it possible for a smaller space to have a larger dimension?
- Can we find a subspace of Fⁿ that doesn't have a basis at all?

(Reduced) row-echelon form

To say A is in **row-echelon form**, or **REF**, means:

- 1. The leftmost entry of each nonzero row of A is 1. (Leading 1s)
- 2. The leading 1s move strictly to the right as we go down the rows of *A*.
- If A is in REF, columns with leading 1s the **pivot columns** of A. If A is in REF, and in addition, all entries *above* every leading 1 are 0, we say that A is in **reduced row-echelon form**, or **RREF**.

Example/picture:

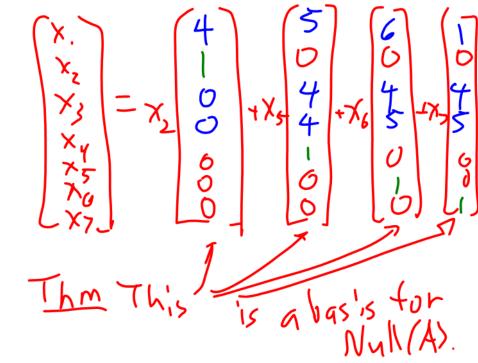
$$A = \begin{bmatrix} 1 & 3 & 0 & 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 & 2 & 2 \end{bmatrix}$$

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Systems in RREF are straightforward to solve With $F = \mathbf{F}_7$, consider the system $A\mathbf{x} = \mathbf{0}$ with matrix $A = \begin{bmatrix} 1 & 3 & 0 & 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 & 2 & 2 \end{bmatrix}$ ars Rewrite equations: +2x+x+6x=0 Xy+3x+2x ・ロト ・ 日 ・ ・ 日 ・ ・ 日 ・ э

= X6 12.10 $= X_{2}$ 6x7 $-2x_e - x_i -$ Χ, Minus Signs! -3×5-3×6-3×1 Xīž - 5×5-7×2-5×2 This tells us that we can $\chi_c =$ choose the free variables χζ freely: x 2, x 5, x 6, x 7 And each such choice $X_{\gamma} =$ gives a unique sol'n. **Thuy.** This process gives a basis for $N_{\rm HH}(A)$ ┍⋍╟

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How can we reduce an arbitrary system to RREF?



Definition

The elementary operations are:

- 1. Switch equation i and equation j \Leftrightarrow switch row i and row j of A.
- 2. For $a \in F$, $a \neq 0$, multiply both sides of equation *i* by $a \Leftrightarrow$ multiply row *i* of A by a.
- 3. For $a \in F$, add a times equation i to equation j \Leftrightarrow add a times row i of A to row j.

Because these operations are reversible, they don't change Null(A).

Gaussian reduction

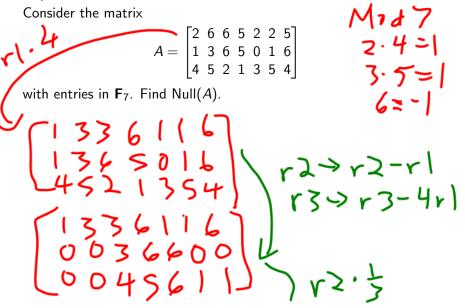
- 1. If A is the $n \times k$ zero matrix, done.
- 2. Else swap rows (type 1) to get leftmost $a \neq 0$ in top row of A.
- 3. Multiply top row by a^{-1} to make leading 1 (type 2).
- 4. Add multiples of the top row to other rows (type 3) to make entries underneath top row leading 1 equal to 0.
- 5. Go back to step 1 and apply Gaussian reduction to the 1 rows of A beneath the top row.

This results in REF.

 \Rightarrow RREF by adding multiples of each nonzero row to the rows above it, right to left, to clear everything above leading 1s. Final result is RREF(A), the **RREF of** A.

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Example



33 6 1167 601 2200 r2-2r U r1+v3 D

[300216] & r1-3r2 0000333 0001322) The RREFweson earlier ! So use earlier work to get basis for Null (A).

The upshot

- Given matrix A, we can compute a basis for Null(A). So we can find a basis for a subspace described as a nullspace (solution set of Ax = 0).
- Given a subspace W of Fⁿ and vectors {v₁,..., v_k} that span W, let A be the matrix with columns v₁,..., v_k. Then {v₁,..., v_k} is linearly independent (and therefore, a basis for W) if and only if Null(A) = 0, i.e., all columns are pivot columns.

Remaining question: If $W = \text{span} \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly dependent, can we reduce $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ to get a basis?

A= [4, - 14]

Contraction

Theorem (Contraction)

The pivot columns of the original matrix A, i.e., columns of A that correspond to pivot columns of RREF(A), form a basis for Col(A). **Idea of proof:**

- Free variables show that non-pivot columns are linear combination of pivot columns, so not needed for span.
- If all free variables = 0, only solution is x = 0, so pivot columns are linearly independent.

Definition

Let A be a $n \times k$ matrix over a field F. We define rank(A), the **rank of** A, to be dim(Col(A)), the dimension of the column space of A, and we define nullity(A), the **nullity of** A, to be dim(Null(A)), the dimension of the nullspace of A. Corollary (Rank-Nullity Theorem)

Corollary (Rank-Nullity Theorem) Let A be a $n \times k$ matrix over a field F. Then rank(A) + nullity(A) = k.

Example Consider the matrix $\begin{bmatrix} 5 & 3 & 1 & 4 & 4 & 4 \\ 2 & 2 & 4 & 6 & 0 & 2 \\ 1 & 3 & 4 & 0 & 5 & 4 \\ 1 & 1 & 2 & 3 & 5 & 6 \end{bmatrix}$ (Vy11(A)= (x)+ 2 ((A)= span (als) with entries in \mathbf{F}_7 . Find bases for Null(A) and Col(A). Ans: Turns out that (0)($= \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\mathsf{RREF}(A)$ basis (01/A) Finding a basis for Null(A) is the same bookkeeping procedure that we saw before.

Theorem (Comparison Theorem)

Let W be a subspace of F^n . If $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ spans W and $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ is a linearly independent subset of W, then $\ell \leq s$.

I.e.: **ANY** linearly independent subset is no larger than **ANY** spanning set.

Why: If $s < \ell$, then we can set up with *s* linear equations in ℓ variables, which must have a nonzero solution. That nonzero solution contradicts linear independence of $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$.

Consequences of Comparison Thm

Corollary (Dimension Theorem)

Any two bases for W must have the same size k (i.e., W cannot have more than one dimension).

Proof:

Corollary

If dim W = k, any linearly independent set must have size $\leq k$ and any span set must have size $\geq k$. **Proof:** PS05.

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So how can we be sure that every subspace has a basis?

Definition

Let W be a subspace of F^n . A maximal linearly independent subset of W is a linearly independent subset $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of Wsuch that for any $\mathbf{x} \in W$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{x}\}$ is linearly dependent.

Theorem

Let W be a subspace of F^n , and suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a maxmimal linearly independent subset of W. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for W.

(proof omitted)

Corollary

If W is a subspace of F^n , then W has a basis.

One more consequence

Corollary (Subspace Size Theorem)

If W is a subspace of a subspace V of F^n , then dim $W \le \dim V \le n$. In particular, any subspace of F^n has dimension at most n.

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