Math 127, Mon Mar 08

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 5.5–5.6 (reload book **again**). 5.5: all new problems!
- Reading for Wed: 6.1–6.2.
- PS04 due tonight; PS05 outline due Wed Mar 10.
- Problem session Fri Mar 12, 10am–noon.

Linear algebra: Questions to resolve

F field

- Given a subspace W of F^n and vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ that span W, how can we check that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for W?
- Given a subspace W of F^n, how can we find a basis for W?
- Is it possible for a subspace W to have one basis with 5 vectors and another basis with 7 vectors? In other words, is it possible for the dimension of W to be both 5 and 7?
- Is it possible for F^8 to contain a subspace of dimension 10? In other words, is it possible for a smaller space to have a larger dimension?
- Can we find a subspace of F^n that doesn't have a basis at all?

(Reduced) row-echelon form

To say A is in **row-echelon form**, or **REF**, means:

1. The leftmost entry of each nonzero row of A is 1. (**Leading 1s**)
2. The leading 1s move strictly to the right as we go down the rows of A.

If A is in REF, columns with leading 1s the **pivot columns** of A.
If A is in REF, and in addition, all entries *above* every leading 1 are 0, we say that A is in **reduced row-echelon form**, or **RREF**.

Example/picture:

pivots

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 & 2 & 2 \end{bmatrix}$$

Systems in RREF are straightforward to solve

With $F = \mathbf{F}_7$, consider the system $A\mathbf{x} = \mathbf{0}$ with matrix

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 & 2 & 2 \end{bmatrix}$$

free vars

Rewrite equations:

$$x_1 + 3x_2 \qquad + 2x_5 + x_6 + 6x_7 = 0$$

$$x_2 = x_2$$

$$x_3 \qquad + 3x_5 + 3x_6 + 3x_7 = 0$$

$$x_4 + 3x_5 + 2x_6 + 2x_7 = 0$$

$$x_5 = x_5$$

Red to RHS:

1 sol'n

$$\begin{aligned}
x_1 &= -3x_2 - 2x_5 - x_6 - 6x_7 \\
x_2 &= x_2 \\
x_3 &= -3x_5 - 3x_6 - 3x_7 \\
x_4 &= -3x_5 - 2x_6 - 2x_7 \\
x_5 &= x_5 \\
x_6 &= x_6 \\
x_7 &= x_7
\end{aligned}$$

Minus signs!

This tells us that we can choose the free variables freely: x_2, x_5, x_6, x_7
And each such choice gives a unique sol'n.

~~**Thm.** This process gives a basis for Null(A).~~

$F = F_7$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ 0 \\ 4 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 6 \\ 0 \\ 4 \\ 5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 1 \\ 0 \\ 4 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>Thm</u> This is a basis for Null(A).

How can we reduce an arbitrary system to RREF?

$A\underline{x} = \underline{0}$

Definition

The **elementary operations** are:

1. Switch equation i and equation j
   $\Leftrightarrow$ switch row i and row j of A.
2. For $a \in F$, $a \neq 0$, multiply both sides of equation i by a
   $\Leftrightarrow$ multiply row i of A by a.
3. For $a \in F$, add a times equation i to equation j
   $\Leftrightarrow$ add a times row i of A to row j.

Because these operations are reversible, they don't change $\mathrm{Null}(A)$.

Gaussian reduction

1. If A is the $n \times k$ zero matrix, done. or $n=0$
2. Else swap rows (type 1) to get leftmost $a \neq 0$ in top row of A.
3. Multiply top row by a^{-1} to make leading 1 (type 2).
4. Add multiples of the top row to other rows (type 3) to make entries underneath top row leading 1 equal to 0.
5. Go back to step 1 and apply Gaussian reduction to the $\cancel{k}\ n - 1$ rows of A beneath the top row.

This results in REF.
$\Rightarrow$ RREF by adding multiples of each nonzero row to the rows above it, right to left, to clear everything above leading 1s.
Final result is RREF(A), the **RREF of** A.

Example

Consider the matrix

$$A = \begin{bmatrix} 2 & 6 & 6 & 5 & 2 & 2 & 5 \\ 1 & 3 & 6 & 5 & 0 & 1 & 6 \\ 4 & 5 & 2 & 1 & 3 & 5 & 4 \end{bmatrix}$$

with entries in $\mathbf{F}_7$. Find Null(A).

Mod 7

$2 \cdot 4 = 1$

$3 \cdot 5 = 1$

$6 = -1$

$r1 \cdot 4$

$$\begin{bmatrix} 1 & 3 & 3 & 6 & 1 & 1 & 6 \\ 1 & 3 & 6 & 5 & 0 & 1 & 6 \\ 4 & 5 & 2 & 1 & 3 & 5 & 4 \end{bmatrix}$$

$r2 \to r2 - r1$

$r3 \to r3 - 4r1$

$$\begin{bmatrix} 1 & 3 & 3 & 6 & 1 & 1 & 6 \\ 0 & 0 & 3 & 6 & 6 & 0 & 0 \\ 0 & 0 & 4 & 5 & 6 & 1 & 1 \end{bmatrix}$$

$r2 \cdot \frac{1}{3}$

$$\begin{bmatrix} 1 & 3 & 3 & 6 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 4 & 5 & 6 & 1 & 1 \end{bmatrix} \xrightarrow{r3-4r2} \begin{bmatrix} 1 & 3 & 3 & 6 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 5 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{2\cdot r3} \begin{bmatrix} 1 & 3 & 3 & 6 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 2 & 2 \end{bmatrix}$$

REF

$$\xrightarrow[r1+r3]{r2-2r3} \begin{bmatrix} 1 & 3 & 3 & 0 & 4 & 3 & 1 \\ 0 & 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 & 2 & 2 \end{bmatrix} \leftarrow r1 - 3r2$$

The RREF we saw earlier!

So use earlier work to get basis for Null(A).

The upshot

- Given matrix A, we can compute a basis for $\mathrm{Null}(A)$. So we can find a basis for a subspace described as a nullspace (solution set of $A\mathbf{x} = \mathbf{0}$).
- Given a subspace W of F^n and vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ that span W, let A be the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent (and therefore, a basis for W) if and only if $\mathrm{Null}(A) = \mathbf{0}$, i.e., all columns are pivot columns.

Remaining question: If $W = \mathrm{span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly dependent, can we reduce $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ to get a basis?

how

$A = [\mathbf{v}_1 \cdots \mathbf{v}_k]$

Contraction

Theorem (Contraction)

The pivot columns of the original matrix A, i.e., columns of A that correspond to pivot columns of $\mathrm{RREF}(A)$, *form a basis for* $\mathrm{Col}(A)$.

Idea of proof:

- **Free variables** show that non-pivot columns are linear combination of pivot columns, so not needed for span.
- If all free variables $= 0$, only solution is $\mathbf{x} = \mathbf{0}$, so pivot columns are linearly independent.

Definition

Let A be a $n \times k$ matrix over a field F. We define $\mathrm{rank}(A)$, the **rank of** A, to be $\dim(\mathrm{Col}(A))$, the dimension of the column space of A, and we define $\mathrm{nullity}(A)$, the **nullity of** A, to be $\dim(\mathrm{Null}(A))$, the dimension of the nullspace of A.

Corollary (Rank-Nullity Theorem)

width

Let A be a $n \times k$ matrix over a field F. Then
$\mathrm{rank}(A) + \mathrm{nullity}(A) = k$.

Example

Consider the matrix

Null(A) = {x | Ax = 0}

Col(A) = span(cols)

$$A = \begin{bmatrix} 5 & 3 & 1 & 4 & 4 & 4 \\ 2 & 2 & 4 & 6 & 0 & 2 \\ 1 & 3 & 4 & 0 & 5 & 4 \\ 1 & 1 & 2 & 3 & 5 & 6 \end{bmatrix}$$

$$\mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 5 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \ldots \right\}$$

with entries in $\mathbf{F}_7$. Find bases for Null(A) and Col(A).

Ans: Turns out that

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathrm{Col}(A) \subseteq \mathbb{F}_7^4$

$\mathrm{Null}(A) \subseteq \mathbb{F}_7^6$

So $\left\{ \begin{bmatrix} 5 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 5 \\ 5 \end{bmatrix} \right\}$ is a basis for Col(A)

Finding a basis for Null(A) is the same bookkeeping procedure that we saw before.

Thank goodness, it all works

Theorem (Comparison Theorem)

Let W be a subspace of F^n. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ spans W and $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ is a linearly independent subset of W, then $\ell \leq s$.

I.e.: **ANY** linearly independent subset is no larger than **ANY** spanning set.

Why: If $s < \ell$, then we can set up with s linear equations in ℓ variables, which must have a nonzero solution. That nonzero solution contradicts linear independence of $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$.

Consequences of Comparison Thm

Corollary (Dimension Theorem)

Any two bases for W must have the same size k (i.e., W cannot have more than one dimension).

Proof:

Corollary

If $\dim W = k$*, any linearly independent set must have size* $\leq k$ *and any span set must have size* $\geq k$*.*

Proof: PS05.

So how can we be sure that every subspace has a basis?

Definition

Let W be a subspace of F^n. A **maximal linearly independent subset of** W is a linearly independent subset $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of W such that for any $\mathbf{x} \in W$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{x}\}$ is linearly dependent.

Theorem

Let W be a subspace of F^n, and suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a maxmimal linearly independent subset of W. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for W.

(proof omitted)

Corollary

If W is a subspace of F^n, then W has a basis.

One more consequence

Corollary (Subspace Size Theorem)

If W is a subspace of a subspace V of F^n, then $\dim W \leq \dim V \leq n$. *In particular, any subspace of F^n has dimension at most n.*