Math 127, Wed Mar 03

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 5.4, 5.5 (reload book again).
- Reading for Mon: 5.6.
- PS04 outline due tonight, full version due Mon Mar 08.
- Problem session Fri Mar 05, 10am–noon.

Recap: Subspaces and linear combinations

F field $F = \mathbb{F}_p, \mathbb{Q}, \mathbb{C}$

Definition

For $n \in \mathbf{N}$, a **subspace** of F^n is $W \subseteq F^n$ s.t.:

1. W contains the zero vector $\mathbf{0}$;
2. (Closed under $+$) For any $\mathbf{v}, \mathbf{w} \in W$, we have $\mathbf{v} + \mathbf{w} \in W$; and
3. (Closed under scalar multiplication) For any $\mathbf{v} \in W$ and $a \in F$, we have $a\mathbf{v} \in W$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in F^n. A **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a vector of the form

$$a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k$$

for $a_i \in F$.

Recap: Spanning and linear independence

$\mathbf{v}_1, \ldots, \mathbf{v}_k$ vectors in F^n, W a subspace of F^n.

Span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is

(n.)

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k \mid a_i \in F\}.$$

(v.) To say $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ **spans** W means both of the following hold:

1. Each of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is contained in W.
2. Every $\mathbf{x} \in W$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

To say $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ **linearly independent** means:

if

$$a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

then

$$\text{all } a_i = 0.$$

Basis, dimension, coordinates

W subspace of F^n.

A **basis** for W is a linearly independent subset $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of W that also spans W.

$\boxed{\dim W} = k$ means that W has a basis with k vectors in it.

Theorem

$\mathcal{B} =$ $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *be a basis for* W. *Then for every* $\mathbf{w} \in W$, *there exists* *unique* $a_1, \dots, a_k \in F$ *s.t.*

$$\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k.$$

$a_1, \dots, a_k$ coords of W w.r.t. $\mathcal{B}$.

Proof: $\boxed{\text{Coords exist}}$.

b/c $\{\underline{v}_1, \dots, \underline{v}_k\}$ spans W, every $w \in W$ is $= a_1\underline{v}_1 + \cdots + a_k\underline{v}_k$ for $a_i \in F$.

$\boxed{\text{Coords unique}}$ S'pose

$$\underline{w} = a_1\underline{v}_1 + \cdots + a_k\underline{v}_k \quad \text{and}$$

$$\underline{w} = b_1\underline{v}_1 + \cdots + b_k\underline{v}_k$$

$$\underline{0} = (a_1-b_1)\underline{v}_1 + \cdots + (a_k-b_k)\underline{v}_k$$

Since $\{\underline{v}_1, \ldots, \underline{v}_k\}$ lin ind,

$$a_1-b_1=0, \ldots, a_k-b_k=0$$

$$\Rightarrow a_1=b_1, \ldots, a_k=b_k$$

Ex of basis, dim.

$F = \mathbb{F}_{11}$

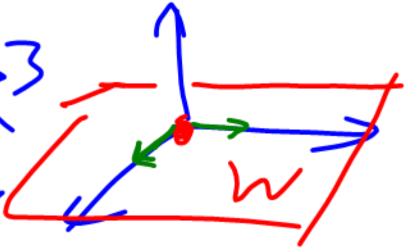
$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; x_1, x_2 \in \mathbb{F}_{11} \right\} \subseteq \mathbb{F}_{11}^5$$

HW

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$ basis for W

$\dim W = 2.$

Ex. $F = \mathbb{R}$; $\mathbb{R}^3$



$W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \middle| \; x, y \in \mathbb{R} \right\}$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ basis for W

$\dim W = 2 \quad (\dim \mathbb{R}^3 = 3)$

The foundations of linear algebra

dimension

basis

span | lin ind

lin comb

$\{\underline{0}\}$ has basis $\emptyset$ (0 vecs) by defn.

($\{\underline{0}\}$ lin dep!)

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \leftarrow \text{lin dep.}$$

$$\underline{v}_1 \qquad \underline{v}_2$$

$$3\underline{v}_1 + 0\underline{v}_2 = \underline{0}$$

What could possibly go wrong? What do we need to compute?

- Is it possible for a subspace W to have one basis with 5 vectors and another basis with 7 vectors? In other words, is it possible for the dimension of W to be both 5 and 7?
- Is it possible for F^8 to contain a subspace of dimension 10? In other words, is it possible for a smaller space to have a larger dimension?
- Can we find a subspace of F^n that doesn't have a basis at all? 4
- Given a subspace W of F^n and vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ that span W, how can we check that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for W?
- Given a subspace W of F^n, how can we find a basis for W?

Answers: Matrices and Gaussian reduction (RREF).

Matrices

An $n \times k$ **matrix over** F is

$\cancel{\mathbb{R}} \Rightarrow F$

$$A = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{bmatrix}}_{k} \left.\vphantom{\begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}}\right\} n .$$

matrix addition

$$A + B = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nk} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1k} + b_{1k} \\ \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & \cdots & a_{nk} + b_{nk} \end{bmatrix}$$

matrix scalar mult

$$cA = c \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{bmatrix} = \begin{bmatrix} ca_{11} & \cdots & ca_{1k} \\ \vdots & \ddots & \vdots \\ ca_{n1} & \cdots & ca_{nk} \end{bmatrix}$$

Row-column and matrix-vector products

$$\underset{(1\times n)}{}\ \underset{\begin{pmatrix}n\\ \times\\ 1\end{pmatrix}}{}$$

$$\mathbf{x}\cdot\mathbf{y} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + \cdots + x_ny_n.$$

row·col is dot prod

If A is $n \times k$, $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$, then

$$A = \begin{bmatrix} \overbrace{-\ r_1\ -}^{k} \\ \vdots \\ -\ r_n\ - \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \mathbf{x} = n\begin{bmatrix} \mathbf{r}_1\cdot\mathbf{x} \\ \vdots \\ \mathbf{r}_n\cdot\mathbf{x} \end{bmatrix}$$

Example:

$$\overbrace{2\underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}}_{4}}^{A}\ 4\overbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}^{x}_{1} = \begin{bmatrix} 1a+2b+3c+4d \\ 5a+6b+7c+8d \end{bmatrix}_{1}$$

$$= a\begin{bmatrix}1\\5\end{bmatrix} + b\begin{bmatrix}2\\6\end{bmatrix} + c\begin{bmatrix}3\\7\end{bmatrix} + d\begin{bmatrix}4\\8\end{bmatrix}$$

= l.c. of cols of A w/ coeffs
a, b, c, d.

An important observation worth its own slide

Suppose:

- A is $n \times k$;
- Columns of A are $\alpha_1, \ldots, \alpha_k$; and
- $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$.

Then

$$A\mathbf{x} = x_1\alpha_1 + \cdots + x_k\alpha_k.$$

$A\mathbf{x}$ is the linear combination of the columns of A with coefficients given by the entries of $\mathbf{x}$.

Matrix multiplication

$$n\,[A]_k \quad k\,[B]_s$$

Suppose:

- A $n \times k$, B $k \times s$;
- $\mathbf{r}_1, \dots, r_n$ the rows of A, $\mathbf{b}_1, \dots, \mathbf{b}_s$ the columns of B.

Then

$$AB = [A\mathbf{b}_1 \ \dots \ A\mathbf{b}_s] = n\underset{s}{\begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b}_1 & \cdots & \mathbf{r}_1 \cdot \mathbf{b}_s \\ \vdots & \ddots & \vdots \\ \mathbf{r}_n \cdot \mathbf{b}_1 & \cdots & \mathbf{r}_n \cdot \mathbf{b}_s \end{bmatrix}}.$$

Can show that matrix multiplication is associative:

$$(AB)C = A(BC)$$

and distributive:

$$A(B+C) = AB + AC \qquad (A+B)C = AC + AB$$

but not commutative.

Nullspace and column space

$$A = n\begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_k \\ | & & | \end{bmatrix} \qquad \mathrm{Col}(A) = \mathrm{Span}\{\underline{v}_1, \ldots, \underline{v}_k\}$$
$$k$$

Suppose A is $n \times k$.

- **Column space of** A, or $\mathrm{Col}(A)$, is defined to be the span of the columns of A, which is therefore a subspace of F^n.
- **Nullspace of** A, or $\mathrm{Null}(A)$, is a subset of F^k defined by

$$\mathrm{Null}(A) = \left\{ \mathbf{x} \in F^k \,\middle|\, A\mathbf{x} = \mathbf{0} \right\}.$$

HW: $\mathrm{Null}(A)$ is a sub**space** of F^k.

The subspaces we'll use will be described either as the span of a given set of vectors (a column space) or the solution set of a system of linear equations (a nullspace).

Back to our important observation

> $A\mathbf{x}$ is the linear combination of the columns of A with coefficients given by the entries of $\mathbf{x}$.

So if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are the columns of a matrix A:

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

$\Leftrightarrow$ The only linear combination of $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ equal to $\mathbf{0}$ is when all coefficients are equal to 0.

$\Leftrightarrow$ The only vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

$\Leftrightarrow$ $\mathrm{Null}(A) = \{\mathbf{0}\}$.

I.e., the columns of A are linearly independent if and only if $\mathrm{Null}(A) = \{\mathbf{0}\}$.

So our computational problems boil down to:

A $n \times k$ over F.

Motivating Problem

Determine if $\mathrm{Null}(A) = \{\mathbf{0}\} \Leftrightarrow$ Columns $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of A are linearly independent.

Motivating Problem

Find a basis for the column space $\mathrm{Col}(A)$.

Motivating Problem

Find a basis for $\mathrm{Null}(A)$.

Answer for all of these: Gaussian elimination/RREF.

Systems of linear equations

$F = \mathbb{F}_p$ or $\mathbb{C}$

System of n **linear equations in** k **variables** is

$$n\left\{\begin{array}{c} a_{11}x_1 + \cdots + a_{1k}x_k = b_1, \\ \vdots \\ \underbrace{a_{n1}x_1 + \cdots + a_{nk}x_k}_{k} = b_n. \end{array}\right. \qquad (1)$$

Note that if

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix},$$

then (1) is $A\mathbf{x} = \mathbf{b}$.

Case $\mathbf{b} = \mathbf{0}$: We say system (1) is **homogeneous**.

Matrix of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is A itself.

(Reduced) row-echelon form

To say A is in **row-echelon form**, or **REF**, means:

1. The leftmost entry of each nonzero row of A is 1. (**Leading 1s**)
2. The leading 1s move strictly to the right as we go down the rows of A.

If A is in REF, columns with leading 1s the **pivot columns** of A. If A is in REF, and in addition, all entries *above* every leading 1 are 0, we say that A is in **reduced row-echelon form**, or **RREF**.

Example/picture:

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 1 & 9 & 10 & 11 \end{bmatrix}$$

A in RREF.

Systems in RREF are straightforward to solve

With $F = \mathbf{F}_{17}$, consider the system $A\mathbf{x} = \mathbf{0}$ with matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 1 & 9 & 10 & 11 \end{bmatrix}$$

Rewrite equations: