Math 127, Spring 2022 **Name:** ________________________

Exam 2

This test consists of 8 questions on 8 pages, totalling 100 points. You are not allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

1. (12 points)

(a) Define what it means for $\mathcal{C}$ to be a binary linear code of length n.

(b) Let $\mathcal{C}$ be a binary linear code of length n. Define what it means for a matrix H to be a parity check matrix of $\mathcal{C}$.

2. (12 points) Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \end{bmatrix},$$

with entries in $\mathbf{F}_3$. Find the RREF of A. Show all your work.

3. (12 points) Let

$$W = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \end{bmatrix} \,\middle|\, x_1, x_4 \in \mathbf{F}_{17} \right\}.$$

You may take it as given that W is a subspace of $\mathbf{F}_{17}^4$.

(a) Guess a basis $\mathcal{B}$ for W. No explanation necessary.

(b) Prove that $\mathcal{B}$ spans W, i.e., prove that every $\mathbf{x} \in W$ is a linear combination of the vectors in $\mathcal{B}$.

4. (12 points) Consider $a(x) = x^5 + x^4 + x^2 + 1$ and $b(x) = x^4 + x^3$ in $\mathbf{F}_2[x]$, and note that

$$\begin{aligned} x^5 + x^4 + x^2 + 1 &= x(x^4 + x^3) + (x^2 + 1), \\ x^4 + x^3 &= (x^2 + x + 1)(x^2 + 1) + (x + 1), \\ x^2 + 1 &= (x + 1)(x + 1). \end{aligned}$$

(I.e., you are given the above facts and do not need to check them yourself.)

Find $f(x), g(x) \in \mathbf{F}_2[x]$ such that $f(x)a(x) + g(x)b(x) = \gcd(a(x), b(x))$. Show your work and clearly indicate your answer.

5. (13 points) Recall that the parity check matrix of the Hamming 7-code $\mathcal{H}_7$ is

$$H_7 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Suppose Yolanda is receiving transmissions sent using the Hamming 7-code, and she receives $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Correct $\mathbf{y}$ to a codeword $\mathbf{y}'$, if necessary, and read off the message bits 3, 5, 6, and 7 to find the intended message $\mathbf{m}'$. Show all your work.

6. (13 points) Let A be a matrix with entries in $\mathbf{F}_7$ such that

$$A = \begin{bmatrix} 3 & 1 & 2 & 0 & 0 \\ 1 & 5 & 2 & 5 & 2 \\ 2 & 3 & 0 & 4 & 4 \\ 1 & 5 & 0 & 0 & 3 \end{bmatrix}, \qquad RREF(A) = \begin{bmatrix} 1 & 5 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find bases for $\mathrm{Col}(A)$ and $\mathrm{Null}(A)$. Show your work.

7. (13 points) Let W be the subset of $\mathbf{F}_7^3$ defined by

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{F}_{19}^3 \,\middle|\, x_1 + 5x_2 = 0 \right\}.$$

Give **part of** the proof that W is a subspace of $\mathbf{F}_{19}^3$, in the following steps:

(a) Explain why $\mathbf{0} \in W$.

(b) Suppose $\mathbf{x}, \mathbf{y} \in W$. Explain why $\mathbf{x} + \mathbf{y} \in W$.

8. (13 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be three vectors in $\mathbf{F}_{11}^4$ and let $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(a) What is the largest possible number of vectors that W could contain? Briefly **EXPLAIN** your answer. (You can just express your answer as a power of some number; you don't have to multiply out that power.)

(b) What condition can you put on the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ that will ensure that W contains that largest possible number of vectors? Briefly **EXPLAIN** your answer.

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