This test consists of 7 questions on 3 pages, totalling 100 points. You are allowed to use one page of notes. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

1. (13 points) Let W be the subset of $\mathbf{F}_{11}^4$ defined by

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbf{F}_{11}^4 \,\middle|\, x_1 + x_2 + x_3 = 0 \right\}.$$

Prove that W is a subspace of $\mathbf{F}_{11}^4$, in the following steps:

(a) Explain why $\mathbf{0} \in W$.

(b) Suppose $\mathbf{x}, \mathbf{y} \in W$. Explain why $\mathbf{x} + \mathbf{y} \in W$.

(c) Suppose $\mathbf{x} \in W$ and $a \in \mathbf{F}_{11}$. Explain why $a\mathbf{x} \in W$.

2. (13 points) **PROOF QUESTION.** Prove that if F is a field, $a, b, c \in F$, $ab = ac$, and $a \neq 0$, then $b = c$.

3. (13 points) Let A be a matrix with entries in $\mathbf{F}_5$ such that

$$A = \begin{bmatrix} 0 & 2 & 4 & 4 & 2 & 1 & 4 \\ 3 & 2 & 0 & 2 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 & 4 & 4 & 1 \\ 3 & 0 & 1 & 4 & 3 & 2 & 1 \\ 3 & 4 & 4 & 3 & 1 & 0 & 1 \end{bmatrix}, \qquad RREF(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 4 & 0 & 1 \\ 0 & 1 & 2 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find bases for $\text{Col}(A)$ and $\text{Null}(A)$. Show your work.

4. (14 points) Recall that the parity check matrix of the Hamming 7-code $\mathcal{H}_7$ is

$$H_7 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Suppose Yolanda is receiving transmissions sent using the Hamming 7-code, and she receives $\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Correct $\mathbf{y}$ to a codeword $\mathbf{y}'$, if necessary, and read off the message bits 3, 5, 6, and 7 to find the intended message $\mathbf{m}'$. Show all your work.

5. (15 points) Consider $a(x) = x^5 + x$ and $b(x) = x^4 + x^3 + x + 1$ in $\mathbf{F}_2[x]$, and note that

$$x^5 + x = (x+1)(x^4 + x^3 + x + 1) + (x^3 + x^2 + x + 1)$$
$$x^4 + x^3 + x + 1 = (x)(x^3 + x^2 + x + 1) + (x^2 + 1)$$
$$x^3 + x^2 + x + 1 = (x+1)(x^2 + 1)$$

(I.e., you are given the above facts and do not need to check them yourself.)

Find $f(x), g(x) \in \mathbf{F}_2[x]$ such that $f(x)a(x) + g(x)b(x) = \gcd(a(x), b(x))$. Show your work and clearly indicate your answer.

6. (16 points) Let $\mathcal{C}$ be the binary linear code of length 7 with parity check matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find a basis for $\mathcal{C}$. Show your work.

(b) What is the dimension of $\mathcal{C}$? Briefly **EXPLAIN** your answer.

(c) How many codewords are there in $\mathcal{C}$? Briefly **EXPLAIN** your answer.

7. (16 points) Consider the subspace

$$W = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \mathbf{v} \right\}$$

of $\mathbf{F}_7^5$, where $\mathbf{v}$ is a vector in $\mathbf{F}_7^5$ to be chosen. Note that the dimension of W may depend on the choice of $\mathbf{v}$.

(a) Find a particular choice of $\mathbf{v}$ such that $\dim W = 3$.
Briefly **JUSTIFY** your answer.

(b) Find a particular choice of $\mathbf{v}$ such that $\dim W \neq 3$.
Briefly **JUSTIFY** your answer.