**Math 127, Spring 2020** **Name:** ______________________
**Exam 1**

This test consists of 8 questions on 5 pages, totalling 100 points. You are not allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

**1.** (12 points) Let $R$ be a ring.

(a) Define what it means for $R$ to have the Zero Factor Property.

(b) Now suppose $R$ has the Zero Factor Property (i.e., $R$ is an integral domain), and suppose $f(x), g(x) \in R[x]$. State the formula for the degree of the product $f(x)g(x)$, i.e., the formula whose left-hand side is $\deg(f(x)g(x))$.

**2.** (12 points)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $5^n \pmod{11}$ | | | | | | | | | | |

(a) Fill in the above table, making sure each entry is calculated in $\mathbf{Z}/(11)$ and is between 0 and 10.

(b) Is 5 a primitive element of $\mathbf{Z}/(11)$? Briefly **EXPLAIN** your answer.

For questions 3–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

**3.** (12 points) **TRUE/FALSE:** Let $F$ be a field, let $f(x) \in F[x]$ be a polynomial, and let $\alpha$ be an element of $F$. If $f(\alpha) = 0$, then it must be the case that $(x - \alpha)$ divides $f(x)$.

**4.** (12 points) **TRUE/FALSE:** Every nonzero element of $\mathbf{Z}/(14)$ is a unit.

**5.** (13 points) Use the Signed Euclidean Algorithm to find $\gcd(105, 31)$. Show all your work. (If you don't know/remember how to use the Signed Euclidean Algorithm, you can use the unsigned Euclidean Algorithm for partial credit.)

**6.** (13 points) Find the multiplicative inverse of 22 in $\mathbf{Z}/(53)$. Show all your work.

**7.** (13 points) Consider the polynomials $f(x) = x^4 - x^3 + x^2 + x + 1$ and $g(x) = x^3 - x^2$ in $\mathbf{F}_3[x]$. Find $\gcd(f(x), g(x))$ in $\mathbf{F}_3[x]$. Show all your work, making sure that at each stage of your calculation, you express each coefficient as 0, 1, or $-1$.

**8.** (13 points) Let $F$ be a field.

(a) To multiply two polynomials of degree 2, we do:

$$\begin{array}{rrrrr} & & a_2x^2+ & a_1x+ & a_0 \\ & \times & b_2x^2+ & b_1x+ & b_0 \\ \hline & & a_2b_0x^2 + & a_1b_0x + & a_0b_0 \\ & a_2b_1x^3 + & a_1b_1x^2 + & a_0b_1x & \\ a_2b_2x^4 + & a_1b_2x^3 + & a_0b_2x^2 & & \\ \hline c_4x^4+ & c_3x^3+ & c_2x^2+ & c_1x+ & c_0 \end{array}$$

(Each $c_i$ is the sum of the elements of $F$ above it.) How many multiplications does this calculation require, and how many additions?

(b) To multiply two polynomials of degree 3, we do:

$$\begin{array}{rrrrrrr} & & & a_3x^3+ & a_2x^2+ & a_1x+ & a_0 \\ & & \times & b_3x^3+ & b_2x^2+ & b_1x+ & b_0 \\ \hline & & & a_3b_0x^3 + & a_2b_0x^2 + & a_1b_0x + & a_0b_0 \\ & & a_3b_1x^4 + & a_2b_1x^3 + & a_1b_1x^2 + & a_0b_1x & \\ & a_3b_2x^5 + & a_2b_2x^4 + & a_1b_2x^3 + & a_0b_2x^2 & & \\ a_3b_3x^6 + & a_2b_2x^4 + & a_1b_2x^3 + & a_0b_2x^2 & & & \\ \hline c_6x^6+ & c_5x^5+ & c_4x^4+ & c_3x^3+ & c_2x^2+ & c_1x+ & c_0 \end{array}$$

How many multiplications does this calculation require, and how many additions?

(c) Assuming the above pattern holds, how many multiplications and how many additions are required to multiply two polynomials of degree $n$ in $F[x]$? Give a big-O estimate of the total time required for this calculation.