Sample questions for Final Exam
Math 126, Spring 2015

Our class has now diverged significantly from what I have done in previous classes, so this sample exam is merely a guideline and should not be considered to be representative in either content or style.

1. (12 points) Define the function $\pi(x)$ and state the Prime Number Theorem.

2. (10 points) Find positive integers a, b, c such that $a^2 + b^2 = c^2$, $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$, a is odd, and $a > 50$. Show all your work.

3. (10 points) Note that $165 = 3 \cdot 5 \cdot 11$. Find an integer x such that $x^{27} \equiv 2 \pmod{165}$. Show all your work.

4. (12 points) Find $g = \gcd(99, 73)$, and find some $x, y \in \mathbf{Z}$ such that $99x + 73y = g$. Show all your work.

5. (12 points) Starting from $14^2 + 13^2 = 5(73)$, use Fermat's Descent Procedure to write the prime 73 as the sum of two squares. Show all your work.

6. (12 points) Does there exist an integer x such that $x^2 \equiv 51 \pmod{71}$? Show all your work, and if you use Quadratic Reciprocity, the QR Multiplication Rule, or other facts about Legendre or Jacobi symbols, justify each step (e.g., "because $37 \equiv 1 \pmod 4$").

For questions 7–12, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

7. (12 points) It is possible that for some odd prime q, $2^{2q} - 1$ is prime.

8. (12 points) (You may take it as given that 1009 is prime.)

For any integer a such that 1009 does not divide a, it must be the case that $a^{1008} - 1$ is divisible by 1009.

9. (12 points) For any positive integers m, n, it must be the case that $\varphi(mn) = \varphi(m)\varphi(n)$ (where φ is the Euler phi function).

10. (12 points) For any integer a such that $\gcd(a, 38) = 1$, it must be the case that $a^{37} \equiv 1 \pmod{38}$.

11. (12 points) For any odd integers $a, m > 1$ such that m does not divide a, it must be the case that there exists some integer x such that $ax \equiv 2 \pmod{m}$.

12. (12 points) There exist infinitely many integers n such that $12n + 5$ is prime.

(cont.)

13. (12 points) **PROOF QUESTION.** Let x, y, k be positive integers such that

$$x^2 + y = 6^k$$

and

$$y \equiv 0 \pmod{15}.$$

Prove that 3 divides x.

14. (12 points) **PROOF QUESTION.** Let f be a multiplicative function such that for any nonnegative integer k,

$$f(2^k) = 3^k, \qquad f(5^k) = \begin{cases} 1 & \text{if } k = 0, \\ 7 & \text{if } k > 0. \end{cases}$$

Suppose n is a positive integer that is a multiple of 10.

(a) Under the above assumptions, what can you say about the prime factorization of n?

(b) Prove that under the above assumptions, $f(n) \equiv 0 \pmod{21}$.

15. (12 points) **PROOF QUESTION.** Let p be an odd prime, let b be an integer, and suppose that p divides $b^2 + 2$. Prove that either $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$.