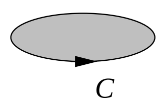
What makes a domain simply connected?
Math 112

Definition. A region $D \subset \mathbb{R}^3$ is said to be *simply connected* if **every** closed curve in D is the boundary of some surface contained **completely within** D. Equivalently, D is simply connected if every closed curve in D can be smoothly contracted (shrunk) to a point without leaving D.

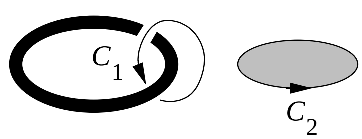
Note that if we carefully take the negation of this definition, we see that:

> To say that a region $D \subset \mathbb{R}^3$ is not simply connected means that there is **at least one** closed curve in D that is not the boundary of **any** surface contained completely within D. Equivalently, to say that D is not simply connected means that there is **at least one** closed curve in D that cannot be smoothly contracted (shrunk) to a point without leaving D.

Example. Let $D = \mathbb{R}^3$, i.e., let D be all of 3-space. Then any closed curve in D is the boundary of some surface in D; namely, take the curve and then "color it in," as shown in the picture below. Therefore, $D = \mathbb{R}^3$ is simply connected.



Example. Let $D = \mathbb{R}^3 - $(a ring-shaped object), i.e., let D be $\mathbb{R}^3$ minus the Aerobie. Then if C_1 is a closed curve that "loops through" the Aerobie, as shown below, it is impossible to find a surface S completely contained within D whose boundary is C_1. Note that the most natural surface whose boundary is C_1, namely, the surface you get by "coloring in" C_1, must cross the Aerobie, and therefore, must leave D. The fact that no other surface has boundary equal to C_1 is also true, but again, far from obvious.



Note also that many closed curves in D are the boundary of some surface in D. For example, the closed curve C_2 is the boundary of a surface in D, namely, the disk on the "inside" of C_2. Nevertheless, because there is one curve C_1 that is not the boundary of a surface in D, D is not simply connected.

Example. Let $D = \mathbb{R}^3 - \{(0,0,0)\}$. Then if C is a closed curve in D, it is possible to find some surface S whose boundary is C. It is certainly not obvious that this is the case, as we must carefully choose S to avoid $(0,0,0)$; however, it can be proven to be true. For example, for a curve C that circles around the origin, as shown below, we may take S to be a sort of "cap" that avoids the origin, but still has boundary equal to C.