Notes on limits and continuity
Math 112

In this handout, we take another look at limits that may make the differences between the one- and two-variable situations a little clearer. To be somewhat formal for a moment, the definition of limit in terms of sequences is:

Definition. We say that a sequence $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots)$ in $\mathbf{R}^n$ **approaches** a point $\mathbf{a}$ in $\mathbf{R}^n$ if, given any neighborhood $D_r(\mathbf{a})$ of $\mathbf{a}$, $\mathbf{x}$ is eventually contained in $D_r(\mathbf{a})$ (i.e., all but many points of $\mathbf{x}$ are contained in $D_r(\mathbf{a})$).

Definition. The function $f : A \subseteq \mathbf{R}^2 \to \mathbf{R}$ has a **limit** L at the point (a, b), written

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if, for **any** sequence (x, y) in A that approaches, but never reaches, (a, b) along **any** route/path, the value of $f(x, y)$ must approach L. In other words, to say that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ means that if $(x, y) \to (a, b)$ with (x, y) never equal to (a, b), then we must have $f(x, y) \to L$.

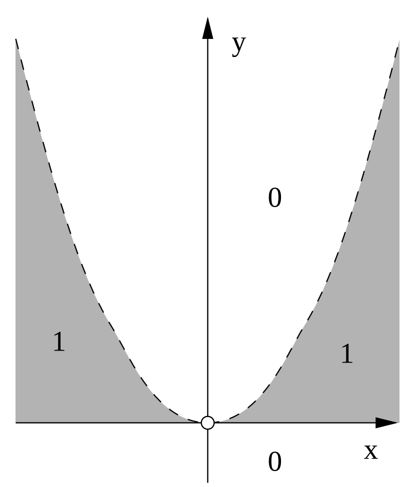
The technical aspects of the term "approach" aside, hopefully the main idea is clear: For $f(x, y)$ to have a limit L at (a, b), the values of $f(x, y)$ must approach L as (x, y) approaches (but does not reach) (a, b), *no matter how* (x, y) *approaches* (a, b). Note that it follows that if we can find at least one way to approach (a, b) along which the values of f do not approach L, then f cannot have a limit of L at (a, b).

Here's an important example to keep in mind. Let f be defined by:

$$f(x, y) = \begin{cases} 1 & \text{if } 0 < y < x^2; \\ 0 & \text{otherwise.} \end{cases}$$

Note that the inequalities in the definition of f are all "less than" and not "less than or equal to."

The function f is perhaps best understood from its contour diagram:



Here, the shaded region is where $f(x, y) = 1$, and the unshaded region (including the curve $y = x^2$ and the entire x-axis) is where $f(x, y) = 0$.

Now, since $f(0, 0) = 0$, if f were continuous at $(0, 0)$, then $f(x, y)$ would have to approach 0 as (x, y) approaches the origin by any route. However, if (x, y) approaches the origin by a route that stays in the shaded region (e.g., along the curve $y = x^2/2$), then $f(x, y)$ remains constant at 1 and therefore does not approach 0. It follows that f is not continuous at $(0, 0)$. In fact, since we may also approach the origin by a route along which $f(x, y) = 0$ (e.g., along the y-axis), we can actually conclude that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, as such a limit would have to be both 0 and 1.

On the other hand, note that if we approach the origin along *any* straight-line path, then the value of $f(x, y)$ will eventually become 0 at some distance away from the origin (why?), so if we only consider straight-line approaches, it looks like $f(x, y)$ is continuous. Again, the fundamental point is that to understand a function of two variables at a point (a, b), it is not enough to look at $f(x, y)$ as (x, y) approaches (a, b) along (for instance) straight lines; you have to look at what happens to $f(x, y)$ as (x, y) approaches (a, b) in an *arbitrary* (straight, curved, spiral, etc.) way. Later on in Chapter 2, we will see that this is precisely the limitation of looking at multivariable functions one variable at a time: namely, that change can occur along paths that cannot be understood by considering only one variable at a time. To repeat:

It is not enough to consider one variable at a time.