The fundamental theorems of calculus
Math 112

All of the fundamental theorems of calculus (e.g., in 3 variables, the fundamental theorem of calculus for line integrals, Stokes' theorem, and the divergence theorem) have the following form:

Theorem. *Suppose that:*

- *M is a space of dimension m;*
- *∂M, the boundary of M, is a space of dimension $m-1$;*
- *ω is a function (possibly a vector function) defined everywhere on M, including its boundary ∂M;*
- *$d\omega$ is a suitable type of derivative of ω (e.g., grad, curl, div).*

Then

$$\int_M d\omega = \int_{\partial M} \omega.$$

For functions of 1 variable, the theorem is just the usual fundamental theorem of calculus:

Name	M	∂M	ω	$d\omega$	Formula
FTC	Interval $[a,b]$	Endpts a,b	$f(x)$	$\frac{df}{dx}$	$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$

For functions of 2 variables, the theorems are the fundamental theorem of calculus for line integrals and Green's theorem:

Name	M	∂M	ω	$d\omega$	Formula
FTC/line	Curve C	Endpts P,Q	$f(x,y)$	$\operatorname{grad} f$	$\int_C (\operatorname{grad} f) \cdot d\mathbf{s} = f(Q) - f(P)$
Green's	2-D region R	Curve C	$\mathbf{F}(x,y)$	$\operatorname{curl} \mathbf{F}$	$\int_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA = \int_C \mathbf{F} \cdot d\mathbf{s}$

The derivatives involved in these theorems form the following chain:

$$\text{Scalar function} \xrightarrow{\text{grad}} \text{Vector field} \xrightarrow{\text{scalar curl}} \text{Scalar function}$$

Note that the composition of two consecutive derivatives in this chain is 0, i.e., $\operatorname{curl}(\operatorname{grad} f) = 0$.

For functions of 3 variables, the theorems are the fundamental theorem of calculus for line integrals, Stokes' theorem, and the divergence theorem:

Name	M	∂M	ω	$d\omega$	Formula
FTC/line	Curve C	Endpts P,Q	$f(x,y,z)$	$\operatorname{grad} f$	$\int_C (\operatorname{grad} f) \cdot d\mathbf{s} = f(Q) - f(P)$
Stokes'	Surface S	Curve C	$\mathbf{F}(x,y,z)$	$\operatorname{curl} \mathbf{F}$	$\int_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{s}$
Divergence	3-D region R	Surface S	$\mathbf{F}(x,y,z)$	$\operatorname{div} \mathbf{F}$	$\int_R (\operatorname{div} \mathbf{F}) \cdot dV = \int_S \mathbf{F} \cdot d\mathbf{S}$

The derivatives involved in these theorems form the following chain:

$$\text{Scalar function} \xrightarrow{\text{grad}} \text{Vector field} \xrightarrow{\text{curl}} \text{Vector field} \xrightarrow{\text{div}} \text{Scalar function}$$

Note that composition of two consecutive derivatives in this chain is 0, i.e., $\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$ and $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.