Some definitions and theorems about conservative vector fields
Math 112

This sheet summarizes some of the main definitions and theorems that arise when trying to solve the problem of determining which vector fields $\mathbf{F}(x, y, z)$ are gradient vector fields. Besides just knowing these facts, the second main point is to understand the distinction between definitions and theorems.

Definition. A vector field $\mathbf{F}(x, y, z)$ is a *gradient*, or *conservative*, vector field if $\mathbf{F}(x, y, z) = \operatorname{grad} f(x, y, z)$ for some differentiable function $f(x, y, z)$.

Definition. A vector field $\mathbf{F}(x, y, z)$ is *path-independent* if $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of the path C.

Definition. A vector field $\mathbf{F}(x, y, z)$ is *circulation-free* if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any *closed* path C.

Theorem. *The following conditions are equivalent. (In other words, 1 implies 2, 2 implies 1, 2 implies 3, and so on.)*

1. $\mathbf{F}(x, y, z)$ *is a gradient vector field.*
2. $\mathbf{F}(x, y, z)$ *is path-independent.*
3. $\mathbf{F}(x, y, z)$ *is circulation-free.*

Theorem. *If* $\mathbf{F}(x, y, z)$ *is a gradient vector field, then* $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

The converse of the last theorem is almost, but not quite, true:

Theorem. *If* $\operatorname{curl} \mathbf{F} = \mathbf{0}$ *and the domain* R *of* $\mathbf{F}$ *is simply connected (every simple closed path in* R *is the boundary of some surface completely contained in* R*), then* $\mathbf{F}(x, y, z)$ *is circulation-free.*

The assumption that R is simply connected is crucial. For a 2-D example, consider the "magic swirly" $\mathbf{F}(x, y) = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}$, whose domain is $\mathbb{R}^2 - (0, 0)$. For a 3-D example, consider a similar magic swirly whose domain is the complement of the Aerobie.