Sample Exam 3
Math 108, Spring 2016

1. (14 points) Let A be a set.

(a) Define what it means for A to be finite. Express your definition in terms of a bijection or bijections.

(b) Define what it means for A to be **in**finite. Express your definition in terms of a bijection or bijections.

In questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (12 points) Let X and Y be sets, and let $f : X \to Y$ and $g : Y \to X$ be functions such that $g \circ f = \mathrm{id}_X$ (the identity function on X). Then it must be the case that $f(g(y)) = y$ for all $y \in Y$.

3. (12 points) Let (x_n) be a sequence such that $\lim_{n\to\infty} x_n = 13$. Then it is possible that $x_{5n} = 14$ for all $n \geq 700$.

4. (12 points) It is possible that there exists a sequence (x_n) such that $x_n > 0$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} x_n = 0$.

5. (16 points) **PROOF QUESTION.** Use the definition of the limit to prove that

$$\lim_{n\to\infty} \frac{7n-2}{3n+5} = \frac{7}{3}.$$

6. (16 points) **PROOF QUESTION.** Let $f : X \to Y$ be a function, and let A and B be subsets of X. Prove that $f(A) \backslash f(B) \subseteq f(A \backslash B)$.

7. (18 points) **PROOF QUESTION.** Recall that $\bigcup_{i=1}^{n} A_i$ is defined recursively by

$$\bigcup_{i=1}^{1} A_i = A_1,$$
$$\bigcup_{i=1}^{n+1} A_i = \left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}.$$

For each positive integer i, suppose that A_i is a nonempty subset of $\mathbf{R}$ and M_i is a real number such that for all $x \in A_i$, $x \leq M_i$. (I.e., suppose each A_i is bounded above.)

Use induction to prove that, for any $n \geq 1$, there exists some real number U_n (possibly depending on n) such that for all $x \in \bigcup_{i=1}^{n} A_i$, $x \leq U_n$. (I.e., prove $\bigcup_{i=1}^{n} A_i$ is bounded above).