Sample Final Exam
Math 108, Spring 2016

1. (10 points) Let A and B be sets.

(a) Define $A \cup B$.

(b) Define $A \setminus B$.

2. (10 points) Let X be a set and let $\sim$ be an equivalence relation on X. For $x \in X$, define E_x, the equivalence class of x under $\sim$.

3. (10 points) Let (x_n) be a real-valued sequence (where $n = 0, 1, \dots$). Define what it means for x_n to be decreasing. (Note that this is "decreasing" and not necessarily "strictly decreasing".)

In question 4–9, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

4. (13 points) **TRUE/FALSE:** Let A be an infinite set. Then it is possible to find a proper subset $S \subset A$ (i.e., a subset $S \subseteq A$ such that $S \neq A$) such that S is also infinite.

5. (13 points) **TRUE/FALSE:** Let $C = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbf{Z}\}$. Then 3 is an element of C.

6. (13 points) **TRUE/FALSE:** Let A and B be sets, and let $f : A \to B$ be a function that is both one-to-one and onto. Then it must be the case that there exists some function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

7. (13 points) **TRUE/FALSE:** Let S be a nonempty bounded subset of $\mathbf{R}$. Then it must be the case that $\inf S \in S$.

8. (13 points) **TRUE/FALSE:** Let X and Y be sets, and let $f : X \to Y$ be a well-defined function. Then it must be the case that for every $y \in Y$, there exists exactly one $x \in X$ such that $f(x) = y$.

9. (13 points) **TRUE/FALSE:** Let either $B = \{1, \dots, n\}$ ($n \in \mathbf{Z}$, $n > 0$) or $B = \mathbf{N}$, and let $f : \mathbf{R} \to B$ be a function. Then it is possible that f is a bijection.

10. (18 points) **PROOF QUESTION.** Recall that the closed interval $[a, b]$ is defined to be $[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$.

Let $g : [-2, 5] \to [3, 28]$ be defined by $g(x) = x^2 + 3$. Prove that g is onto.

11. (18 points) **PROOF QUESTION.** Let A and B be sets, and let $f : A \to B$ be a function. Define a relation $\sim$ on A by declaring that for $a_1, a_2 \in A$,

$$a_1 \sim a_2 \text{ if and only if } f(a_1) = f(a_2).$$

Prove that $\sim$ is an equivalence relation.

12. (18 points) **PROOF QUESTION.** Use induction to prove that for $n \in \mathbf{Z}$, $n \geq 1$, we have that $5^n \geq 3n + 2$.

13. (18 points) **PROOF QUESTION.** Let

$$A = \{n \in \mathbf{Z} \mid n > 0\},$$
$$B = \{n \in \mathbf{Z} \mid n = 6k \text{ for some } k \in \mathbf{Z}\},$$
$$C = \{n \in \mathbf{Z} \mid n \geq 0 \text{ and } n = 3m \text{ for some } m \in \mathbf{Z}\}.$$

(a) Prove that $A \cap B \subseteq C$.

(b) Prove (by example) that $A \cap B \neq C$.

14. (20 points) **PROOF QUESTION.** In the following question, if you cannot prove part (a), go on to part (b), assuming that (a) is true. Note also that once definitions and assumptions are made, they continue to be used for the rest of the problem.

Let A and B be sets, and let $f : A \to \mathbf{Z} \times \mathbf{Q} \times \mathbf{N}$ and $g : \mathbf{Z} \times \mathbf{Q} \times \mathbf{N} \to B$ be functions such that $g \circ f$ is one-to-one.

(a) Use the definition of one-to-one to prove that f is one-to-one.

(b) Prove that A is countable.