Math 108, problem set 06
Outline due: Wed Mar 16
Completed version due: Mon Mar 21
Last revision due: Wed Apr 27

Definitions: Let $f : X \to Y$ be a function, let A be a subset of X, and let B be a subset of Y.

restriction We define the *restriction of f to A* to be the function $f|_A : A \to Y$ defined by $f|_A(x) = f(x)$ for all $x \in A$.

co-restriction If it happens to be the case that for all $x \in X$, we have $f(x) \in B$, we define the *co-restriction of f to B* to be the function $f|^B : X \to B$ given by $f|^B(x) = f(x)$ for all $x \in X$.

bi-restriction If it happens to be the case that for all $x \in A$, we have $f(x) \in B$, we define the *bi-restriction of f to A, B* to be the function $f|_A^B : A \to B$ given by $f|_A^B(x) = f(x)$ for all $x \in A$.

Exercises (to be done but not turned in): 14.1, 14.2, 14.3, 14.4, 14.6, 14.9, 15.4, 15.5, 15.8, 15.9.

Problems to be turned in: All numbers refer to problems in the Yellow and Blue Book. You will also need to use the definition of *composite function* from chapter 16 (p. 167).

1. Define a function $g : \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ by the formula $g(x+y) = (x, y)$ for all $x, y \in \mathbf{Z}$. Is g well-defined? Prove your answer.

2. Complete (in as interesting a manner as possible) and prove the following theorem: Let A, B, and Y be sets, and let $f : A \to Y$ and $g : B \to Y$ be well-defined functions. Then the formula
$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B, \end{cases}$$
gives a well-defined function $h : (A \cup B) \to Y$ if and only if (condition on f and g).

3. Let f and g be functions from $\mathbf{R}$ to $\mathbf{R}$.
   (a) If $f(x) = g(x)$ for infinitely many $x \in \mathbf{R}$, is it necessarily the case that $f = g$? Prove or give a counterexample.
   (b) If $f(x) = g(x)$ for all but finitely many $x \in \mathbf{R}$, is it necessarily the case that $f = g$? Prove or give a counterexample.

4. 14.16.

(Cont. on next page.)

5. (a) 15.19(a).
   (b) For any function $f : X \to Y$ and any $A \subseteq X$, define

$$f(A) = \{y \in Y \mid y = f(a) \text{ for some } a \in A\}\,.$$

   Prove the **Embedding Lemma:** If $f : X \to Y$ is one-to-one and $A \subseteq X$, then the function $g : A \to f(A)$ given by $g(a) = f(a)$ for all $a \in A$ is a bijection.

6. 15.23.

7. (a) 15.26(a).
   (b) 15.26(c). (You may assume the results of 15.26(b) here.)