Math 108
Summary of cardinality

Numbers in parentheses refer to the Yellow and Blue Book.

Definition. Two sets A and B are said to be *equivalent* if there exists a bijection $f : A \to B$. We write $A \approx B$ in that case, and we also say that A and B have the same *cardinality*.

Theorem (Thm. 21.1)**.** *Equivalence is an equivalence relation.*

Lemma (Embedding Lemma)**.** *Let X and Y be sets. If $f : X \to Y$ is one-to-one and $A \subseteq X$, then the function $g : A \to f(A)$ given by $g(a) = f(a)$ for all $a \in A$ is a bijection, and therefore, $A \approx f(A)$.*

Definition. A set A is said to be *finite* if either $A = \emptyset$ or $A \approx \{1, \ldots, n\}$ for some $n \in \mathbf{Z}^+$; otherwise, A is *infinite.*

Theorem (Cor. 21.10)**.** *If S is a finite set, and $A \subseteq S$, then A is finite.*

Theorem (Thm. 21.11)**.** *If A and B are finite sets, then $A \cup B$ is finite.*

Theorem (Cor. 21.14)**.** *If A and B are finite sets, then $A \times B$ is finite.*

Theorem (Thm. 22.2, a.k.a. the Pigeonhole Principle)**.** *If $m, n \in \mathbf{Z}^+$, $m > n$, and $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ is a function, then f is not one-to-one.*

Corollary (Thm. 22.3)**.** *The set $\mathbf{N}$ is infinite.*

Definition. If $A \approx \{1, \ldots, n\}$ for $n \in \mathbf{Z}^+$, we say that $|A| = n$. (This is well-defined only because of the Pigeonhole Principle.)

Definition. A set A is said to be *countably infinite* if $A \approx \mathbf{N}$. A set A is said to be *countable* if A is either finite or countably infinite; otherwise, A is *uncountable.*

Lemma (Ex. 23.5/The Countability Lemma)**.** *Let A be a nonempty set. Then A is countable if and only if there exists an injective function $f : A \to \mathbf{N}$.*

Theorem (Cor. 23.4)**.** *If S is a countable set, and $A \subseteq S$, then A is countable.*

Theorem (Thm. 23.6)**.** *If A and B are countable sets, then $A \cup B$ is countable.*

Theorem (Cor. 23.10)**.** *If A and B are countable sets, then $A \times B$ is countable.*

Theorem (Thm. 23.13)**.** *If A_n is a countable set for each $n \in \mathbf{Z}^+$, then $\bigcup_{n \in \mathbf{Z}^+} A_n$ is a countable set.*

Theorem (Thm. 23.11)**.** *The set of rational numbers, $\mathbf{Q}$, is countably infinite.*

Theorem (Thm. 23.12)**.** *The set of real numbers, $\mathbf{R}$, is uncountable.*

Proof: Cantor diagonalization.