Discrete Isoperimetric Inequalities: Making "Balls" in a World of Dots

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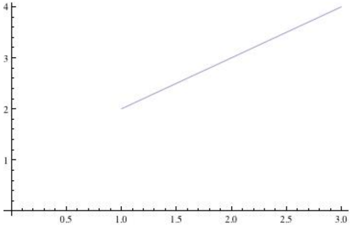
Outline

Metric Spaces

A *Metric Space* (X, d) is a set of points X along with a distance function $d : X \times X \to \mathbb{R}$ such that

- $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

<u>Ex:</u> $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}$ (i.e., each x_i is a real number)$\}$ with the Euclidean ("straight line") distance:

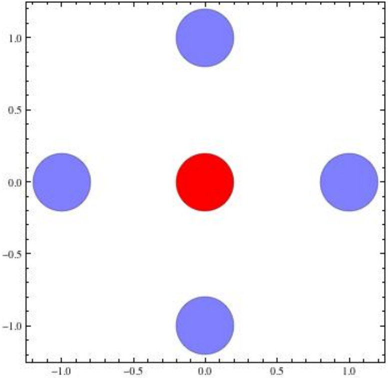


$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Ex: ℓ_1-metric on $\mathbb{R}^n$, otherwise known as "taxicab metric"

$$d_1((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = |x_1 - y_1| + \cdots + |x_n - y_n|$$

$$d_1((1,0),(1,1)) = 1 \qquad d_1((0,1),(1,1)) = 1$$
$$d_1((1,0),(0,1)) = 2 \qquad d_1((0,1),(2,3)) = 4$$



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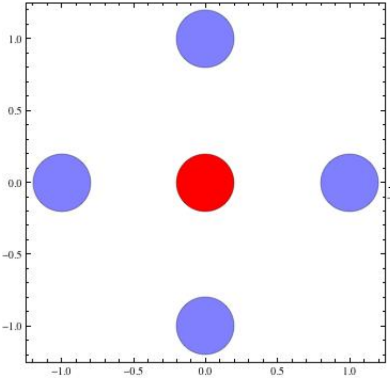

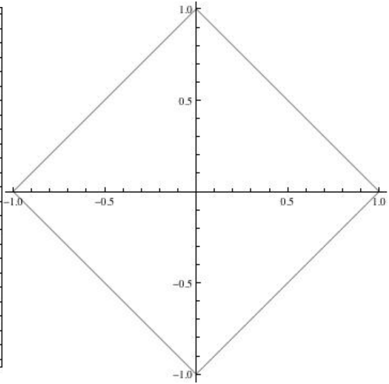

Ex: ℓ_∞-metric on $\mathbb{R}^n$

$$d_\infty((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \max_{i=1,2,\ldots,n}\{|x_i - y_i|\}$$

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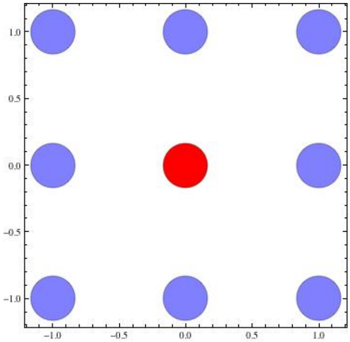

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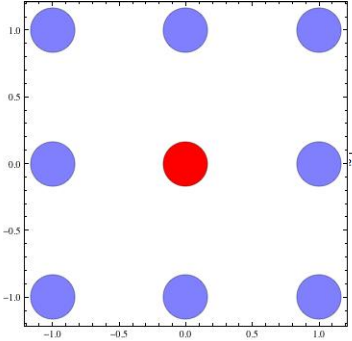

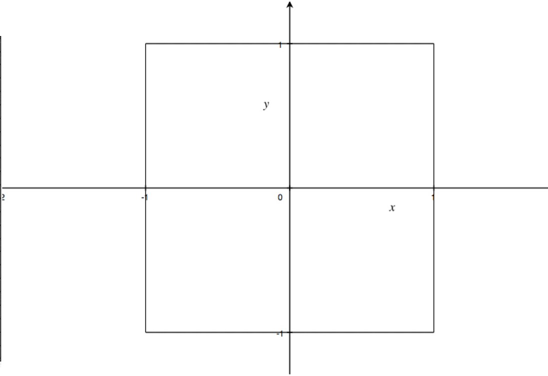

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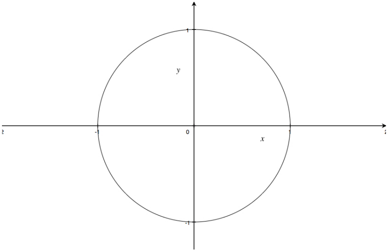

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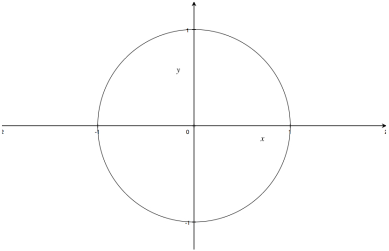
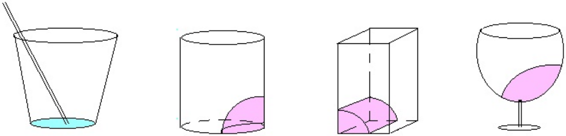




Outline

What is an Isoperimetric Inequaity?

Euclidean Isoperimetry

Discrete Isoperimetry in $\mathbb{Z}^n$

Discrete Isoperimetry With Phase Changes

Euclidean Distance and Boundary

Theorem (The Euclidean Isoperimetric Inequality)

Let $A \subset \mathbb{R}^n$ be a compact set and let $V(A)$ denote the Lebesgue measure of A in $\mathbb{R}^n$. Define

$$b(A) = \lim_{h \to 0^+} \frac{V(A_h) - V(A)}{h}$$

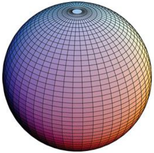
where

$$A_h = \{x \in \mathbb{R}^n : ||x - a||_2 \leq h \text{ for some } a \in A\}$$

Let B_A be the Euclidean ball whose Lebesgue measure is the same as that of A. Then

$$b(A) \geq b(B_A)$$

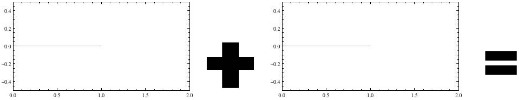
and equality holds true if and only if A is a Euclidean ball.



Fun Proof using Brunn-Minkowski

Define

$$A + B = \{a + b : a \in A, b \in B\}$$



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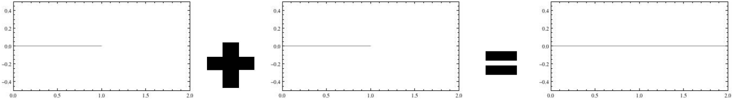

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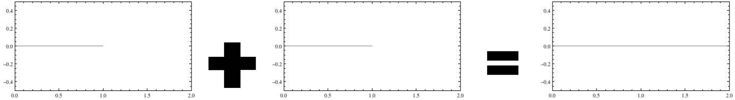

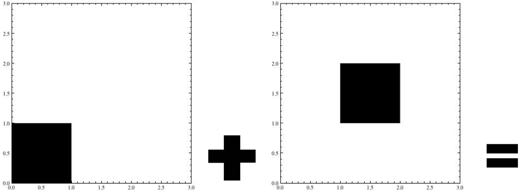

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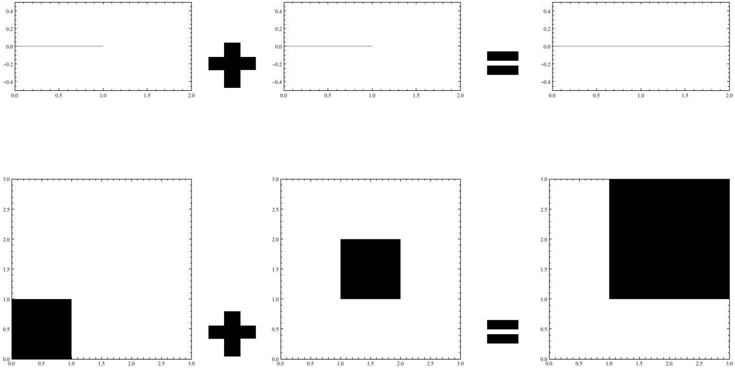
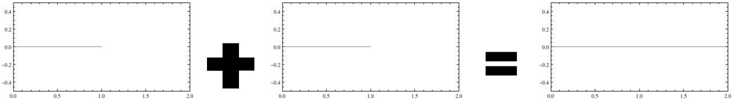


Theorem (Brunn-Minkowski Inequality)

Let $A, B \subset \mathbb{R}^n$ be nonempty, bounded, measurable sets such that $A + B$ is also measurable. Then

$$Vol(A + B)^{1/n} \geq Vol(A)^{1/n} + Vol(B)^{1/n}$$



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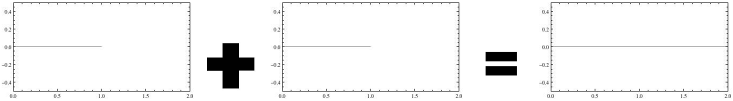

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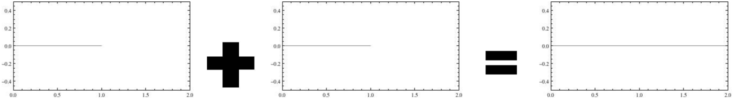

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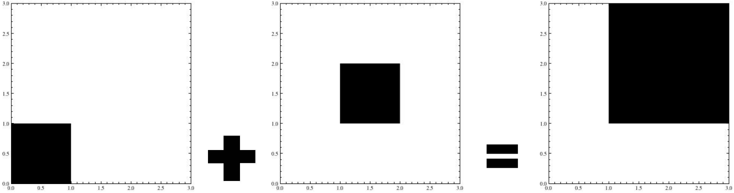

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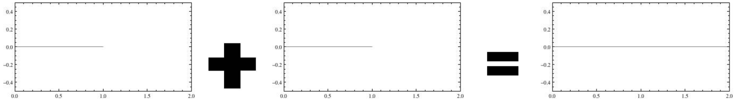

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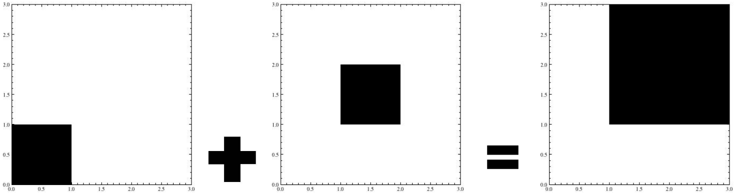
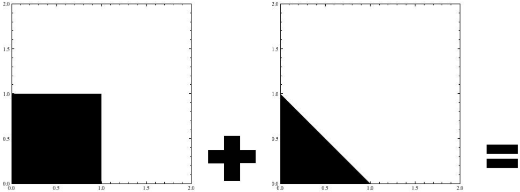


$$4^{1/2} \geq 1^{1/2} + 1^{1/2}$$



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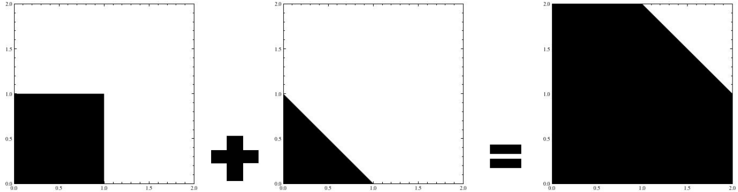
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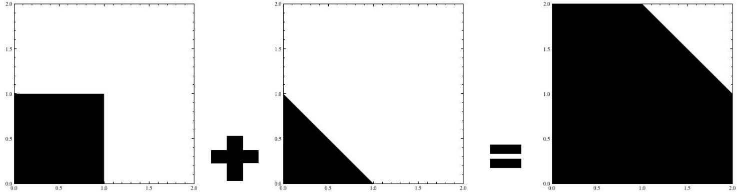

$$\left(\frac{7}{2}\right)^{1/2} \geq 1^{1/2} + \left(\frac{1}{2}\right)^{1/2}$$

$$1.87082869\ldots \geq 1 + 0.707106781\ldots$$

Using Brunn-Minkowski to prove the Isoperimetric Inequality for $\mathbb{R}^n$

Let $A \subset \mathbb{R}^n$, have the same volume as the ball of radius 1 in $\mathbb{R}^n$:

$$B_1 = \{x \in \mathbb{R}^n : d(x, 0) \leq 1\}$$

We would like to show that A has boundary at least as big as the boundary of B_1.

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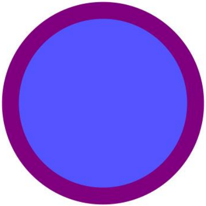
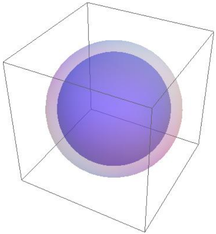
$$A_t = \{x \in \mathbb{R}^n : \text{ for some } a \in A, d(a, x) \leq t\}$$

Then the boundary of A is

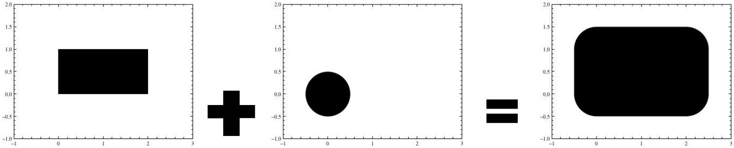
$$\lim_{t \to 0} \frac{\text{Vol}(A_t) - \text{Vol}(A)}{t}$$





Additionally, if we denote
$B_t =$ ball centered at origin of radius $t = \{x \in \mathbb{R}^n : d(x,0) \leq t\}$ then

$$A_t = A + B_t$$



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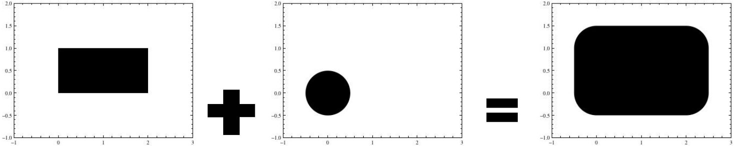


Thus, using Brunn-Minkowski, we have

$$\begin{aligned}
\mathrm{Vol}(A_t)^{1/n} = \mathrm{Vol}(A + B_t)^{1/n} &\geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B_t)^{1/n} \\
&= \mathrm{Vol}(B_1)^{1/n} + \mathrm{Vol}(B_t)^{1/n} \\
&= \mathrm{Vol}(B_1)^{1/n} + \mathrm{Vol}(tB_1)^{1/n} \\
&= \mathrm{Vol}(B_1)^{1/n} + (t^n \mathrm{Vol}(B_1))^{1/n} \\
&= (1 + t)\mathrm{Vol}(B_1)^{1/n} = ((1 + t)^n \mathrm{Vol}(B_1))^{1/n} \\
&= \mathrm{Vol}(B_{1+t})^{1/n} = \mathrm{Vol}((B_1)_t)^{1/n}
\end{aligned}$$

Thus, we have

$$\mathrm{Vol}(A_t) \geq \mathrm{Vol}((B_1)_t)$$

so that

$$\lim_{t \to 0} \frac{\mathrm{Vol}(A_t) - \mathrm{Vol}(A)}{t} \geq \lim_{t \to 0} \frac{\mathrm{Vol}((B_1)_t) - \mathrm{Vol}(B_1)}{t}$$

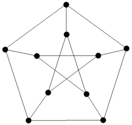
$$\mathrm{BoundaryVolume}(A) \geq \mathrm{BoundaryVolume}(B_1)$$

Thus, for a fixed volume, the ball has the smallest boundary!

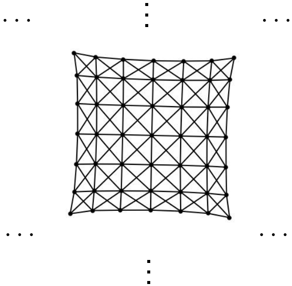
Outline

A graph defined on $\mathbb{Z}^n$

A *graph* is a set of vertices V, along with a set of edges $E \subset V \times V$. We typically visualize a graph with dots for the vertices, arcs for the edges:



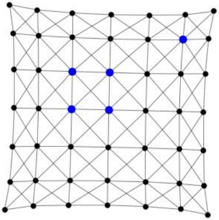
We consider the infinite graph whose vertices are $\mathbb{Z}^n$ and edges are between two points whose ℓ_∞ distance is 1.



Boundary of a subset of $\mathbb{Z}^n$

For $A \subset \mathbb{Z}^n$, we define the boundary of A, ∂A, to be the set of vertices whose distance from A is no more than 1:

$$\partial A = \{x \in \mathbb{Z}^n : d_\infty(x, a) \leq 1 \text{ for some } a \in A\}$$



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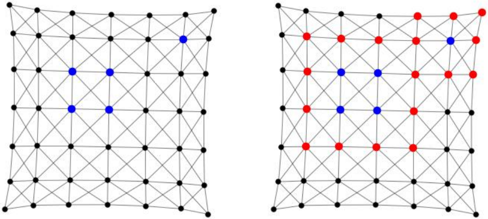
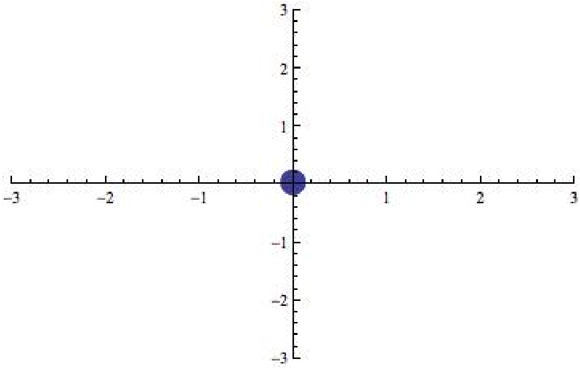


Note: This is slightly different from our earlier discussions of boundary, as this definition implies that $A \subset \partial A$.

Q: What sets of size $1, 2, 3, \ldots$ have the smallest possible boundary?

In $\mathbb{Z}^2$:



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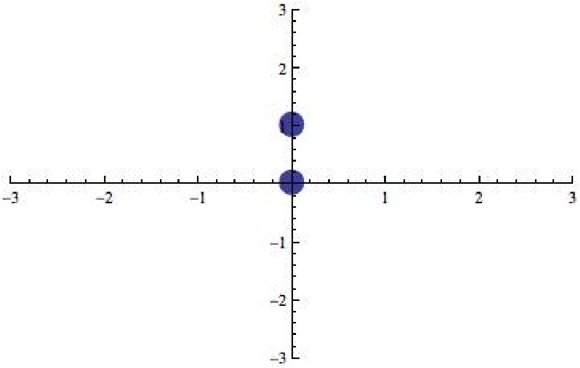

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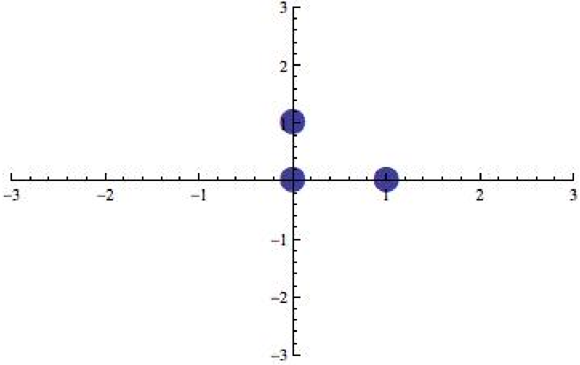

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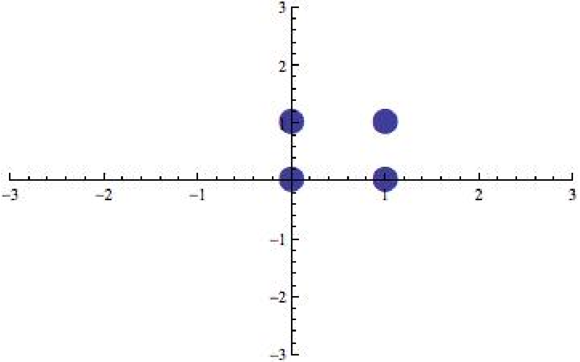

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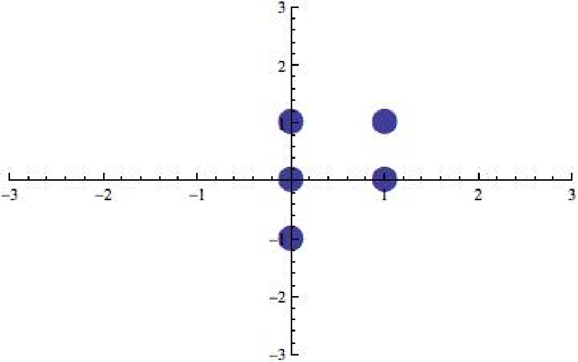

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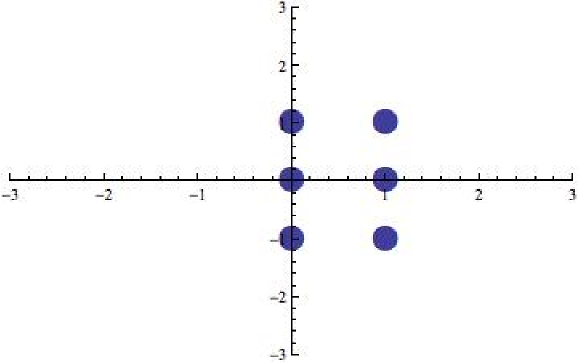

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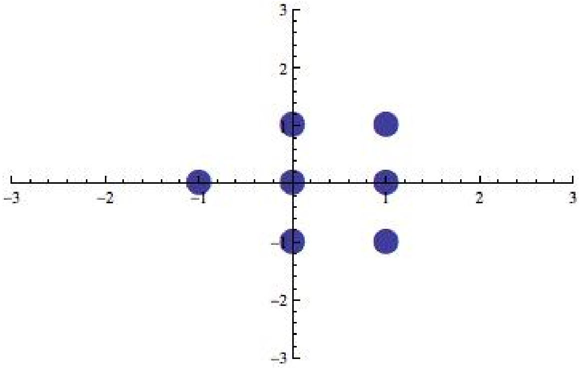

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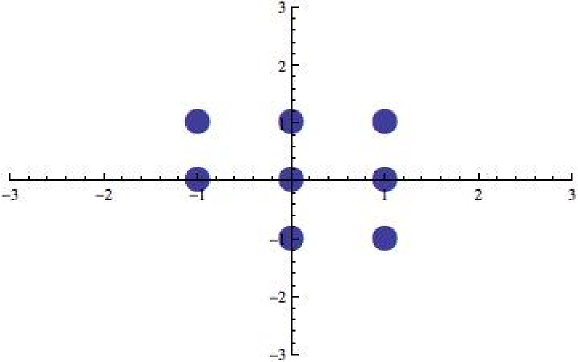

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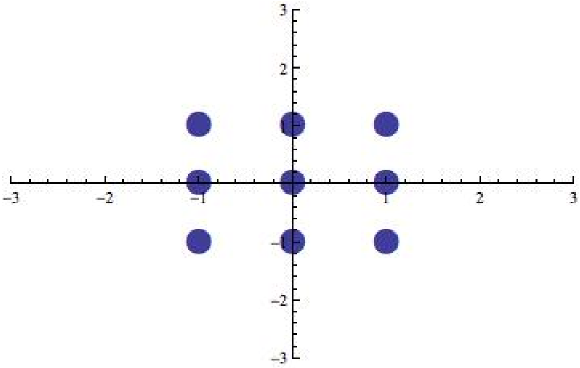

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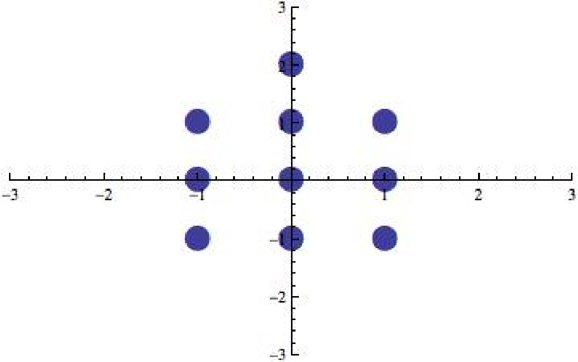

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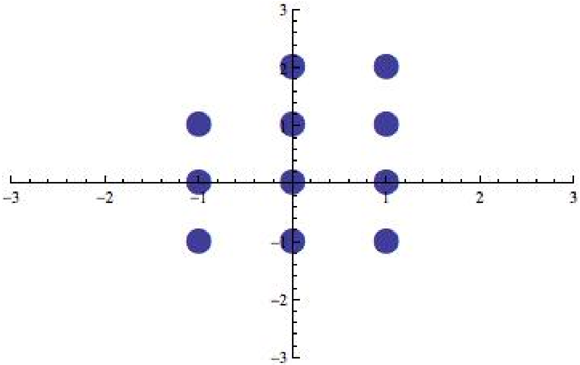

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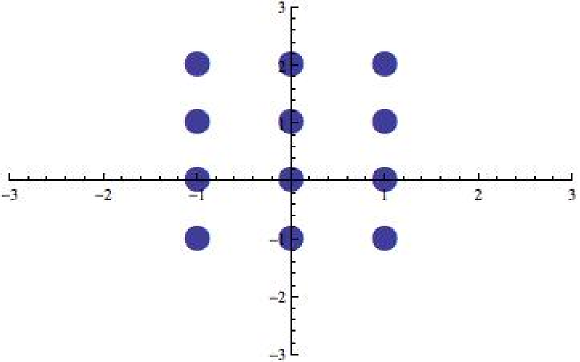

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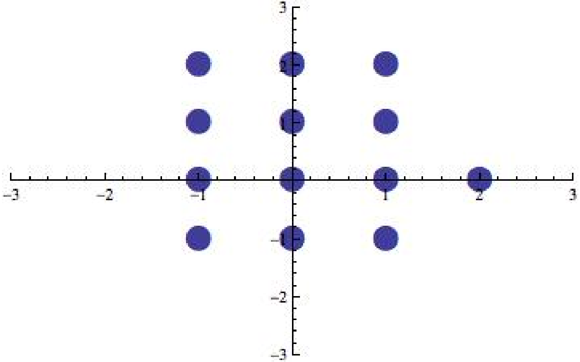

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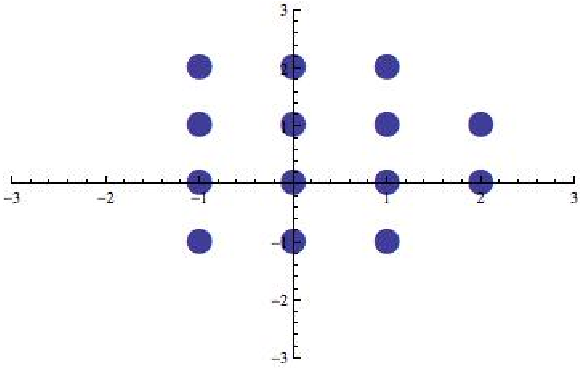

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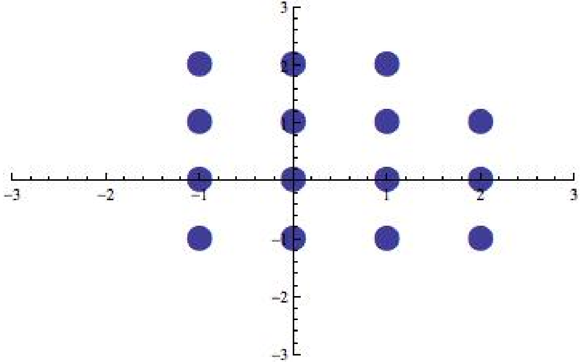

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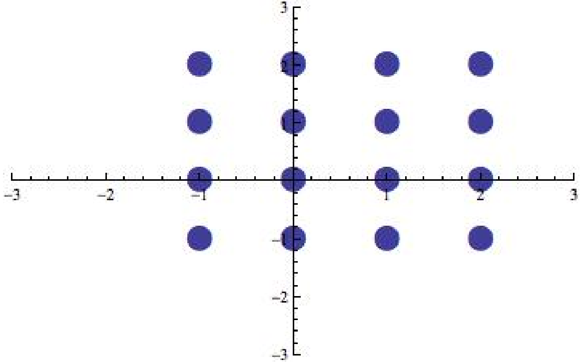

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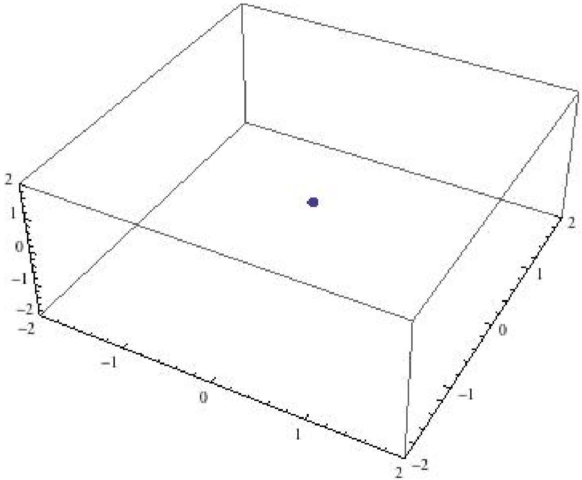

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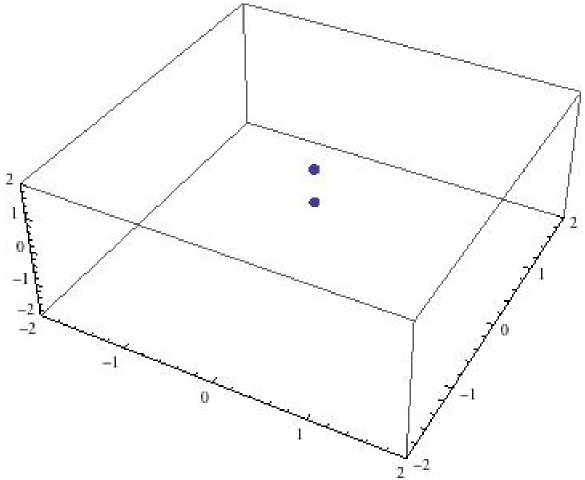

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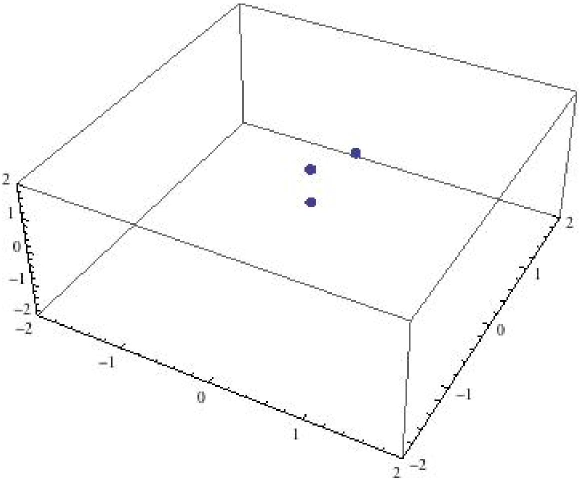

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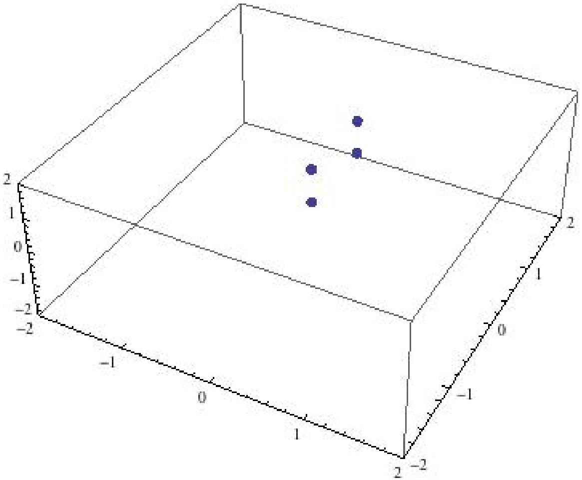

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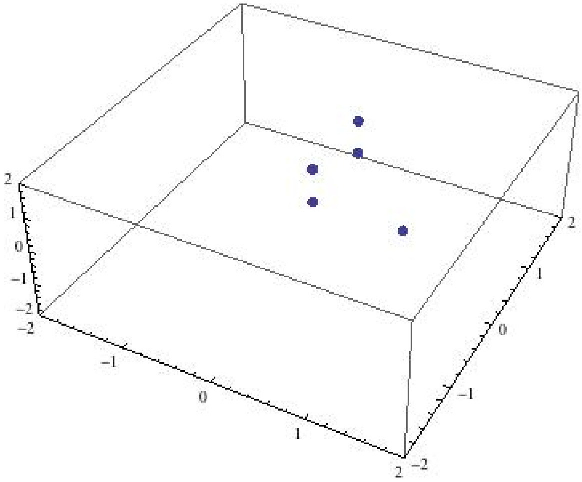

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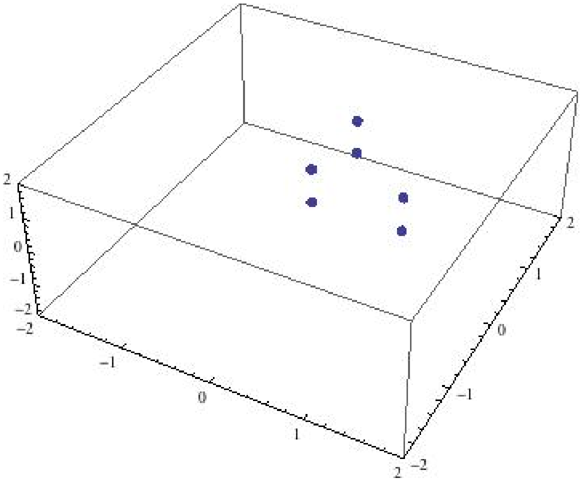

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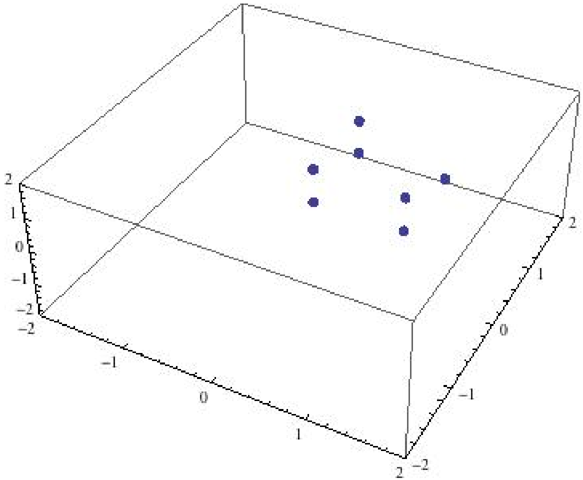

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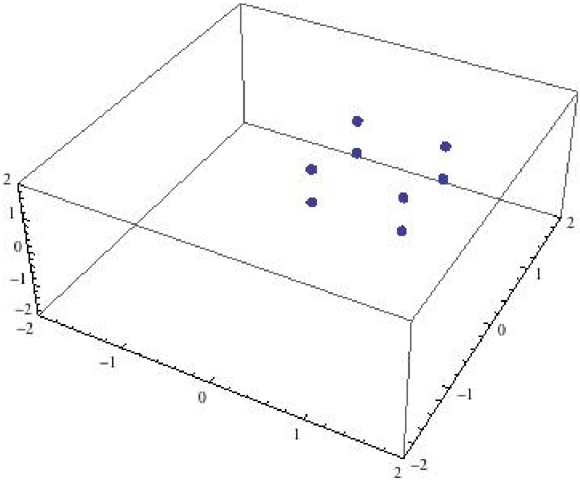

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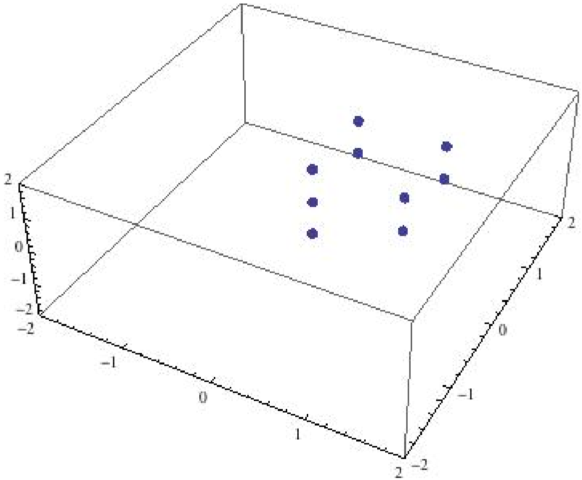

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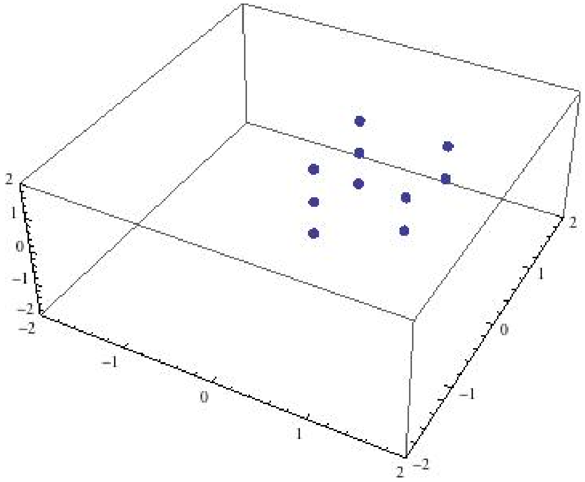

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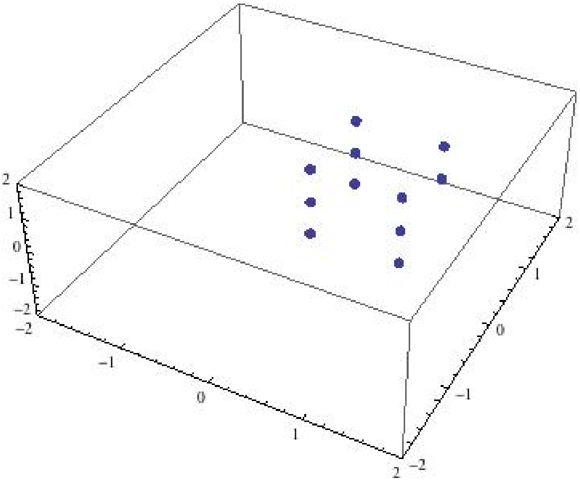

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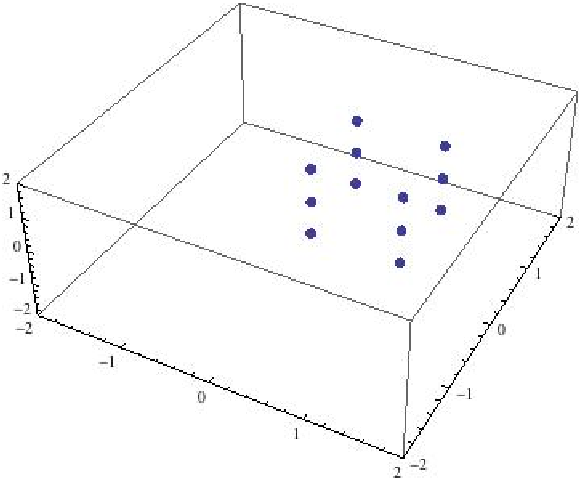

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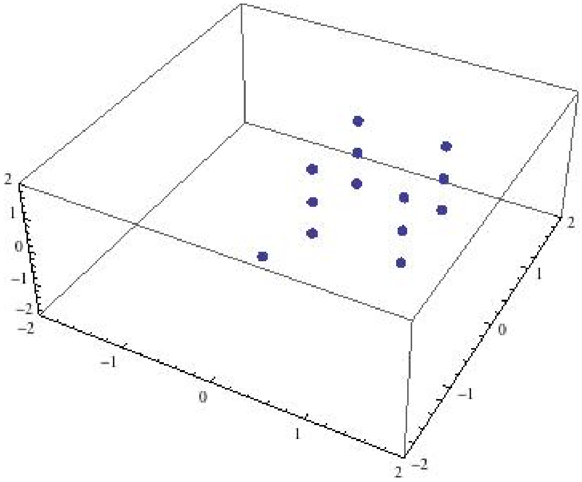

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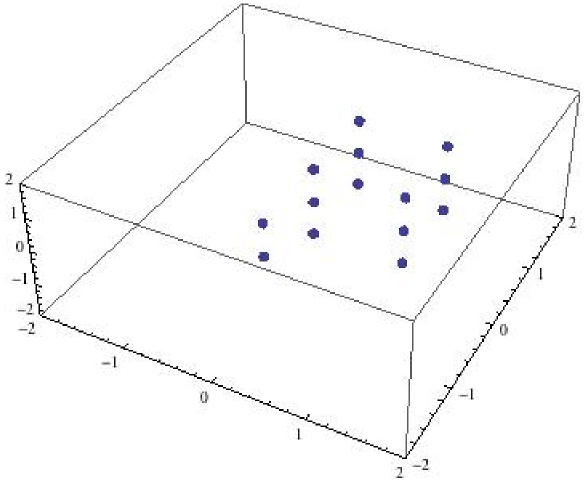

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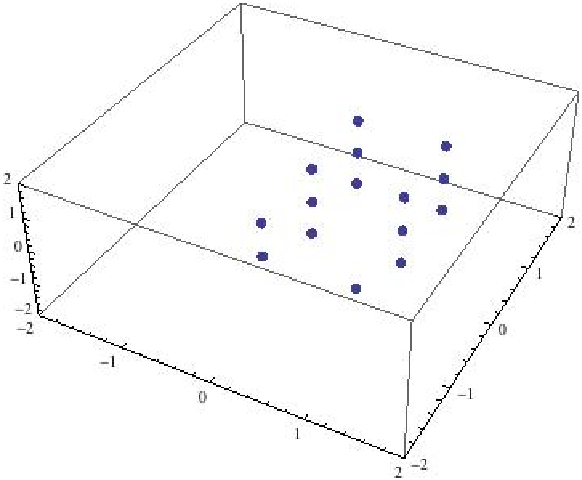

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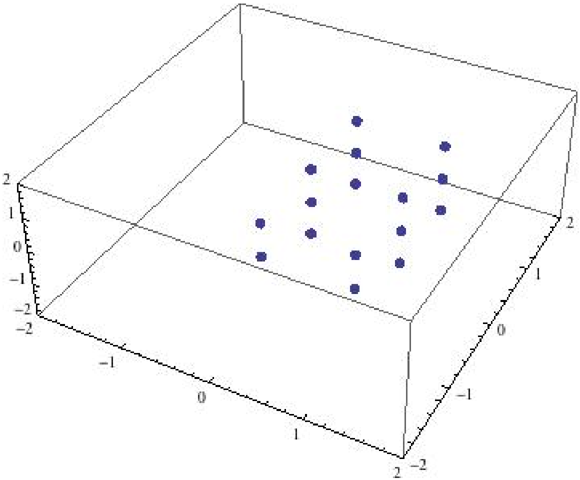

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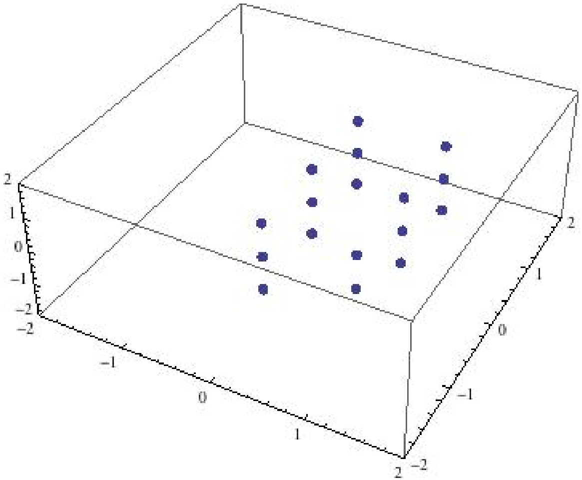

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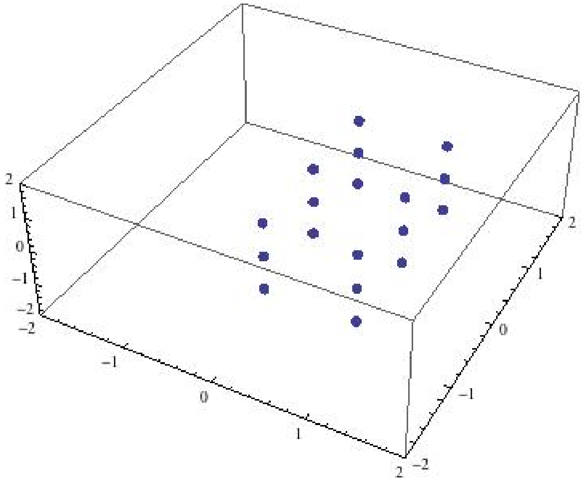

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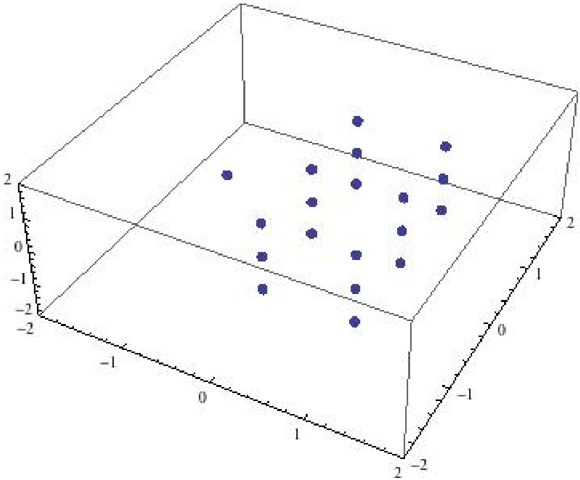

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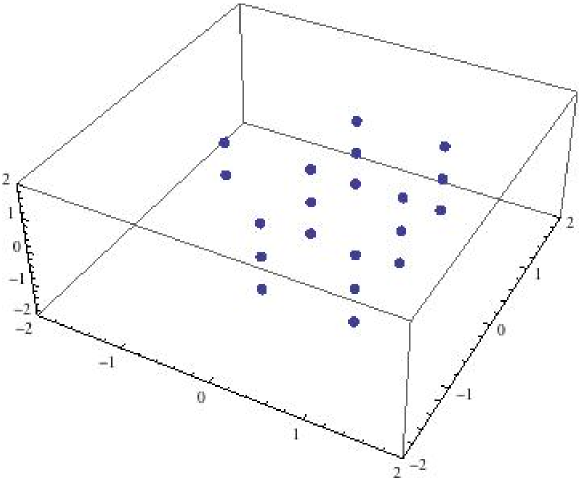

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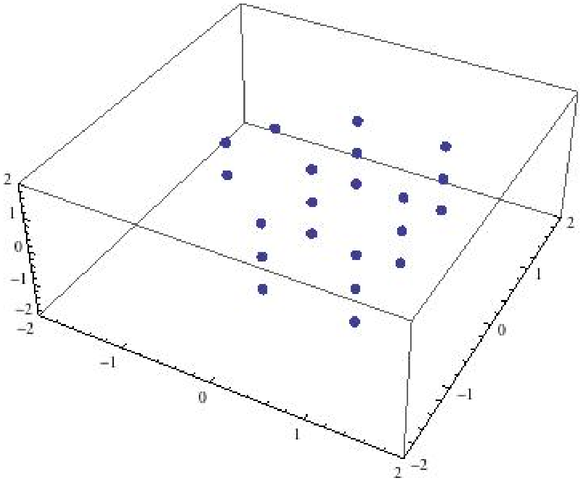

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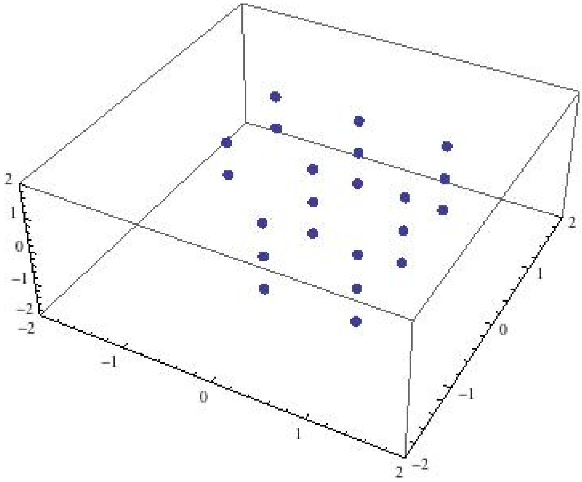

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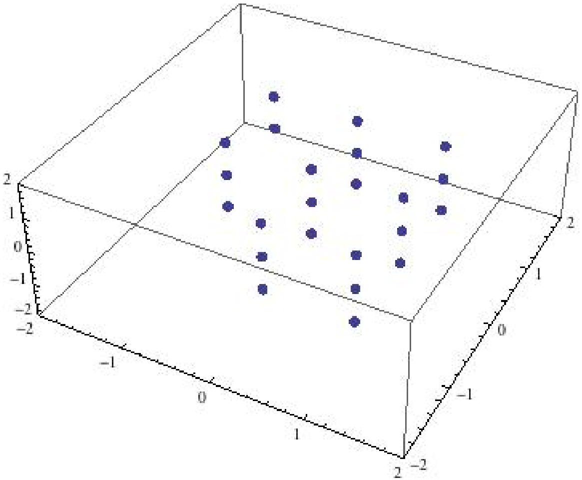

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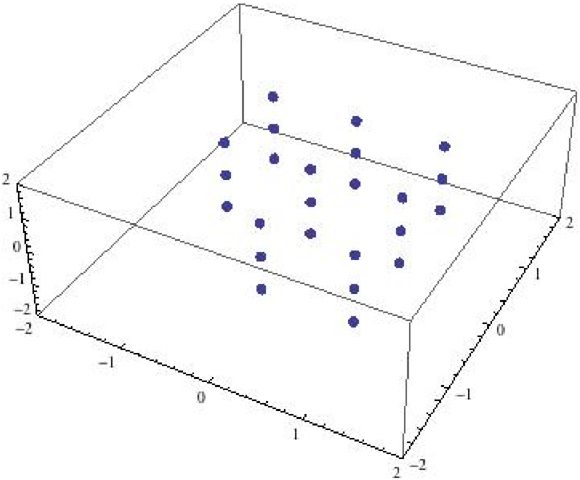

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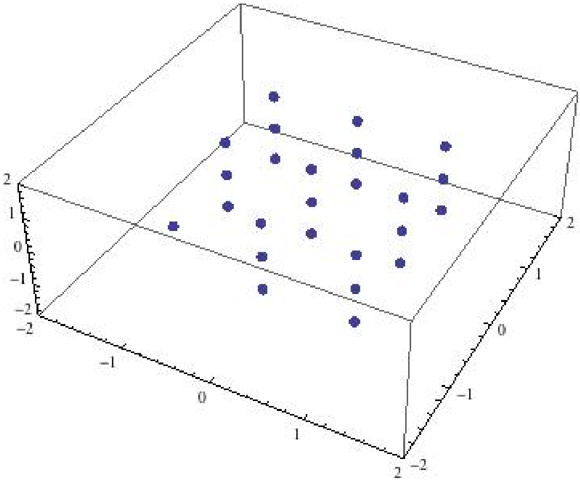

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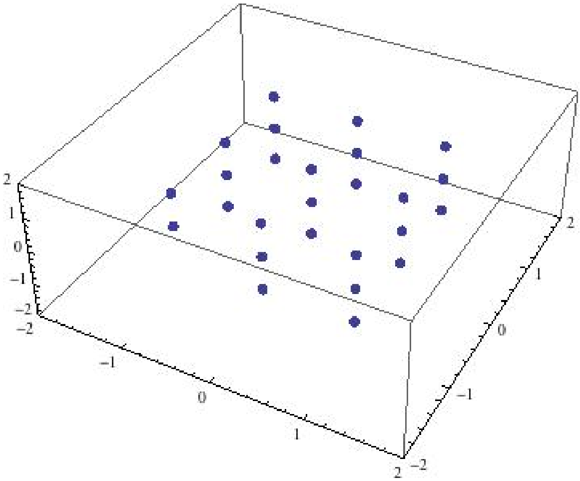

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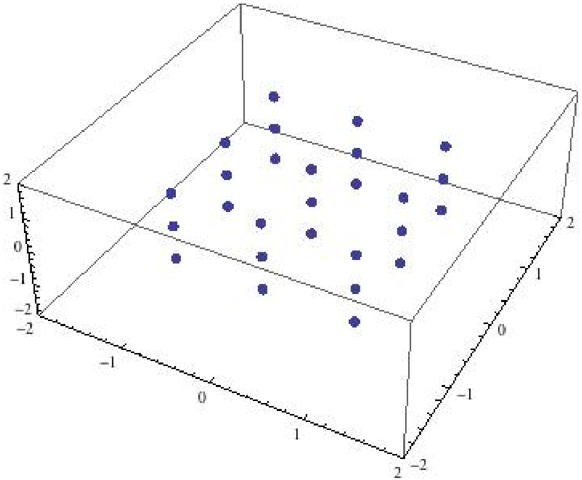

Theorem (V.)

There exists an ordering on the vertices $\mathbb{Z}^n$ such that a set with minimal boundary is achieved at an initial segment.

In other words, suppose we write the elements of $\mathbb{Z}^n$ in a list according to this ordering.

1. (0,0,0)	6.(1,0,1)	11. (1,0,-1)	16. (1,-1,1)
2. (0,0,1)	7. (1,1,0)	12. (1,1,-1)	17. (0,-1,-1)
3. (0,1,0)	8. (1,1,1)	13. (0,-1,0)	18. (1,-1,-1)
4. (0,1,1)	9. (0,0,-1)	14. (0,-1,1)	19. (-1,0,0)
5. (1,0,0)	10. (0,1,-1)	15.(1,-1,0)	20. (-1,0,1)

Define I_k to be the set containing the first k elements in this list. Then for any $A \subset \mathbb{Z}^n$ with $|A| = k$,

$$|\partial I_k| \leq |\partial A|$$

Idea of Proof: Pick any $A \subset \mathbb{Z}^n$ of size k. Pick $i \in \{1, 2, \ldots, n\}$. Look at the "$n-1$-dimensional sections" of A:

$$\begin{aligned} A_{i,0} &= \{(x_1, x_2, \ldots, x_n) \in A : x_i = 0\} \\ A_{i,1} &= \{(x_1, x_2, \ldots, x_n) \in A : x_i = 1\} \\ A_{i,-1} &= \{(x_1, x_2, \ldots, x_n) \in A : x_i = -1\} \\ &\vdots \end{aligned}$$

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For example:
$A = \{(2, 3, 1, 4), (2, -2, -1, 0), (2, 0, -1, 5), (7, 2, 3, -1)\}$
Compressing on the first coordinate:
$\{(2, 0, 0, 0), (2, 0, 0, 1), (2, 0, 1, 0), (7, 0, 0, 0)\}$

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The resulting set has a boundary of no larger size.

Idea of proof, con't.

Continue, compressing with respect to other indices.

If you can't compress any more, you may need to "jiggle" the set a little (not increasing the boundary) so that you can finally compress to an initial segment.

In every step, we didn't increase the boundary, thus the initial segment provides a lower bound on the boundary.

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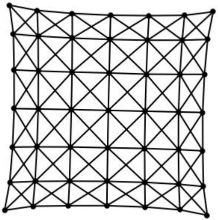
□

Remark: This method of compression is common in the theory of discrete isoperimetric inequalities.

Outline

A Similar Graph

Now we consider the graph $[m]^n = \{0, 1, 2, \ldots, m-1\}^n$ where two vertices are connected by an edge precisely when their d_∞-distance is 1:



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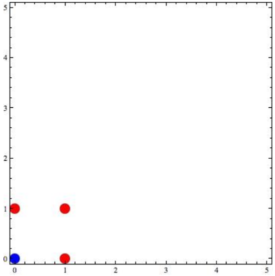

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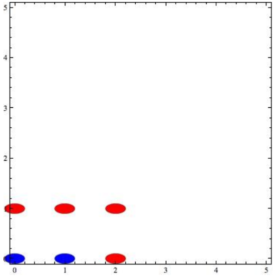

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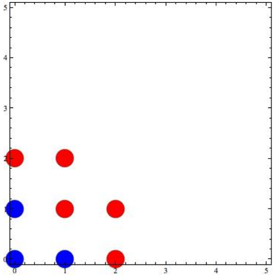

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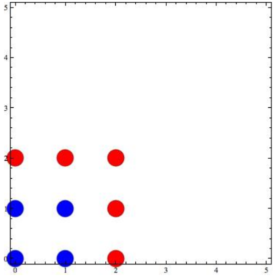

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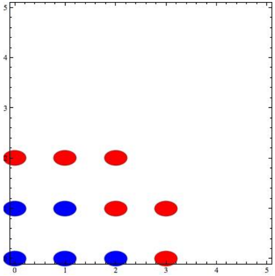

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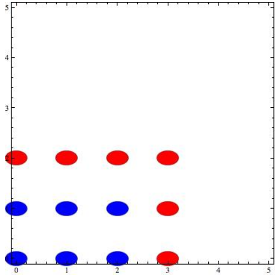

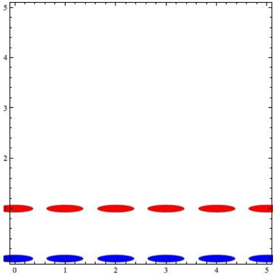

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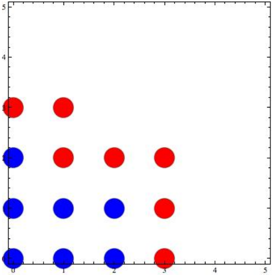

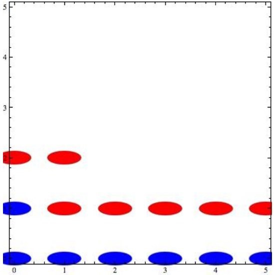

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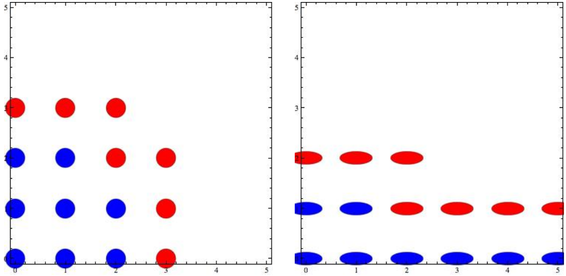

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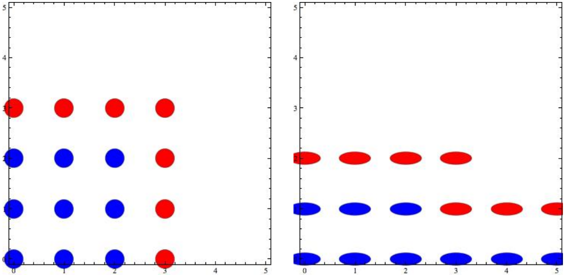

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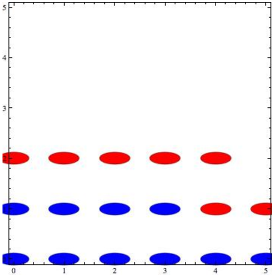

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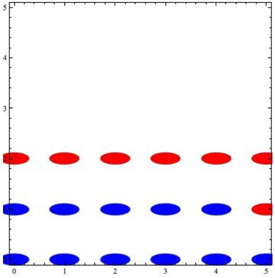

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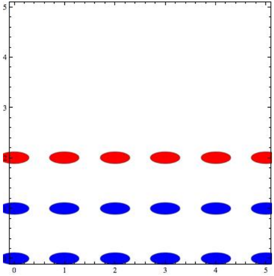

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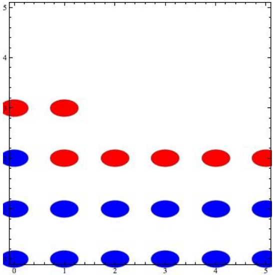

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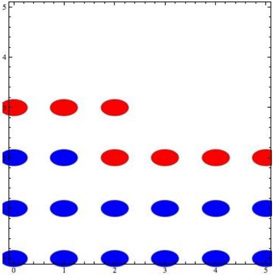

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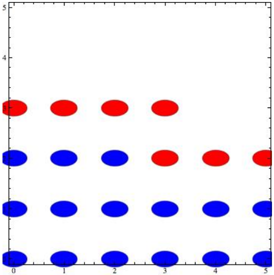

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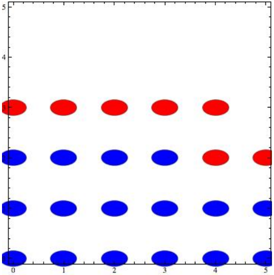

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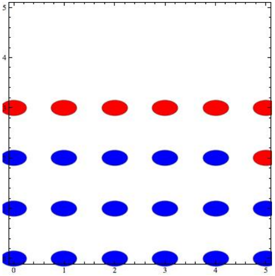

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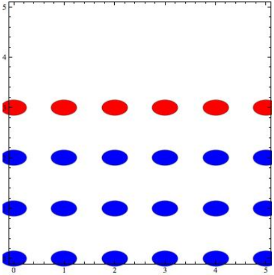

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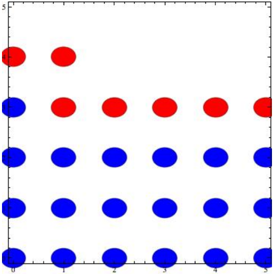

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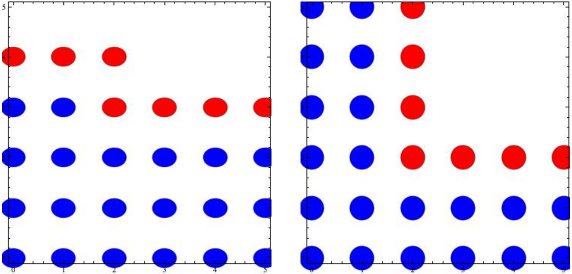

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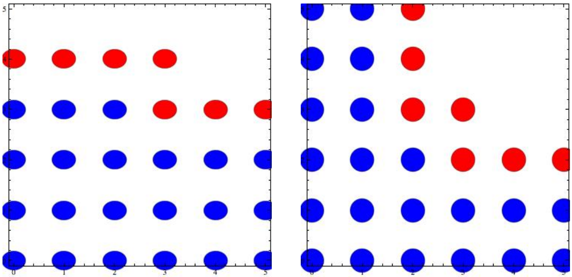

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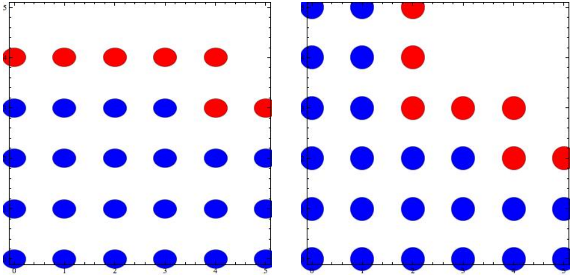

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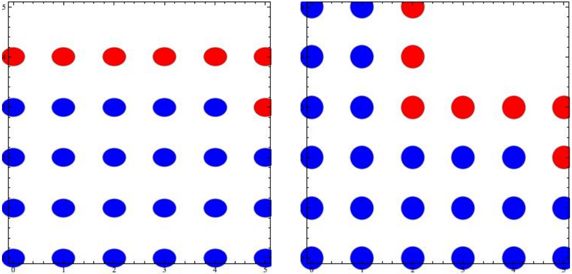

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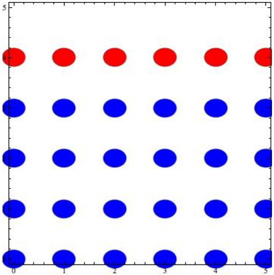

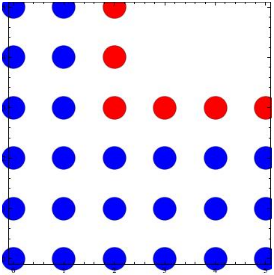

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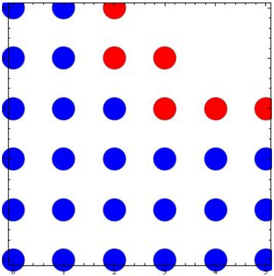

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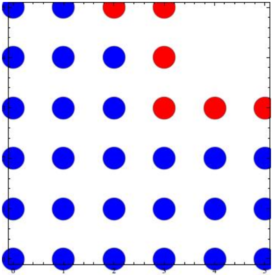

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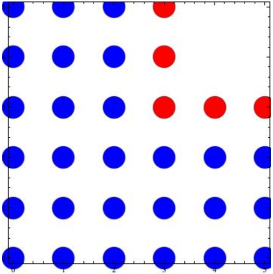
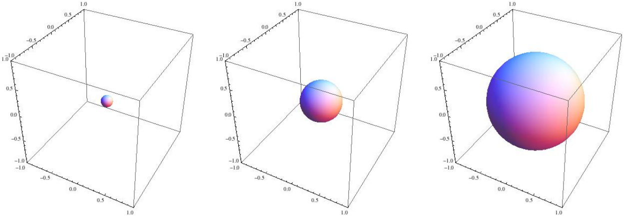
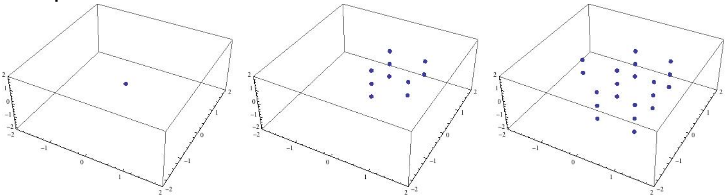


Upshot

Sets of minimal boundary in $\mathbb{R}^n$ with the Eudlidean distance are spheres:

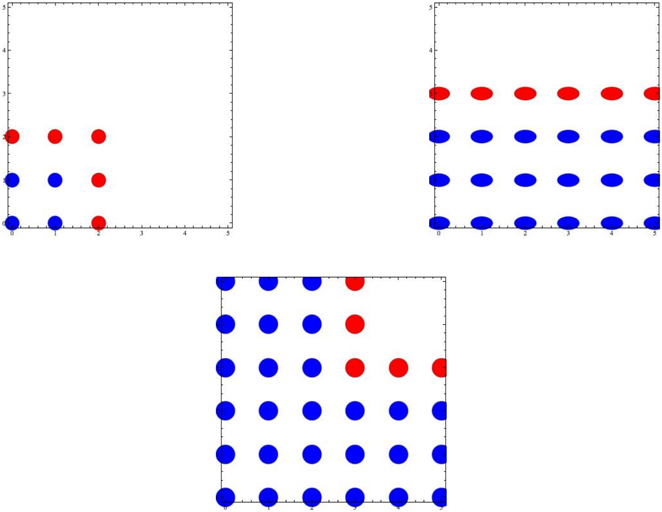


Sets of minimal boundary in $\mathbb{Z}^n$ with the ℓ_∞ distance grow like boxes, and parts of boxes:



Thus, in both of these cases, the sets of minimal boundary are *nested*.

But as we saw for the graph $[6]^2$, sets of minimal boundary *may not be nested*



Open Questions

- For general $[m]^n$ (using the ℓ_∞ metric), what sets have minimum boundary?

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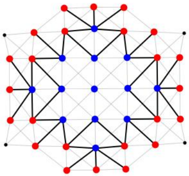
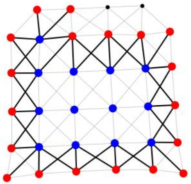
- Define the "edge boundary" of a set to be the number of edges exiting it





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- For $\mathbb{Z}_m^n$ (using the ℓ_∞ metric), what sets have minimum boundary?
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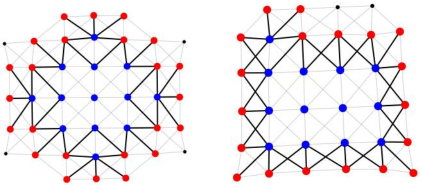

For $\mathbb{Z}_m^n$ (using the ℓ_∞ metric), what sets have minimum edge boundary?