A bijection on core partitions

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Introduction

My research relates to the study of *symmetry*, with applications to Chemistry, Physics, Differential Equations . . . .

In this talk we'll describe some geometry associated to the *symmetric group* of permutations.

Symmetric Group

The *symmetric group* S_n is the set of all **bijections**

$$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots n\}$$

with **composition** as the group operation.

For example, $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{bmatrix}$ and $\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{bmatrix}$ are permutations in S_5 and if we compose them, we get

$$\tau\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{bmatrix}.$$

Symmetric Group

The *symmetric group* S_n has a presentation with generators

$$s_1, s_2, \dots, s_{n-1}$$

and relations

$$s_i^2 = id$$

$$s_i s_j = s_j s_i \qquad \text{for } |i-j| \geq 2,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

We think of each s_i as an *adjacent transposition*:

$$s_i = \begin{bmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i+1 & i & \cdots & n \end{bmatrix}.$$

Symmetric Group

Groups with a similar presentation in terms of generators and relations are called *Coxeter groups*. They have generators $s_1, s_2, \dots, s_n$ with each $s_i^2 = id$, and

$$(s_i s_j)^{m_{ij}} = id.$$

for some $m_{ij} \geq 2$. The relation $s_i^2 = id$ means that $s_i = s_i^{-1}$.

For example,

$$id = (s_i s_j)^2 = s_i s_j s_i s_j$$

is equivalent to

$$s_j s_i = s_i s_j$$

and

$$id = (s_i s_j)^3 = s_i s_j s_i s_j s_i s_j$$

is equivalent to

$$s_j s_i s_j = s_i s_j s_i$$

Symmetric Group

The relations in a Coxeter group are often visualized in a combinatorial graph.

- Vertices = generators.
- No edge $\iff (s_i s_j)^2 = id \iff s_i s_j = s_j s_i$.
- Unlabeled edge $\iff (s_i s_j)^3 = id \iff s_i s_j s_i = s_j s_i s_j$.
- Edge labeled by $m \iff (s_i s_j)^m = id \iff s_i s_j s_i \cdots = s_j s_i s_j \cdots$.

For example, S_n has the Coxeter graph

$$\bullet_{s_1} \text{———} \bullet_{s_2} \text{———} \bullet_{s_3} \quad \cdots \quad \bullet_{s_{n-2}} \text{———} \bullet_{s_{n-1}}$$

Symmetric Group

The relation $s_i^2 = id$ means that $s_i = s_i^{-1}$. We can view the generators s_i as **reflections** of a vector space.

Definition

Let $u \in \mathbb{R}^n$. A *reflection* through the hyperplane orthogonal to u is a linear map s_u sending

$$v \mapsto v - \frac{2\langle u, v\rangle}{\langle u, u\rangle} u$$

Symmetric Group

For S_n, we can combine our two points of view if we take a vector space $\mathbb{R}^n$ with orthonormal basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$.
E.g. let $n = 3$. Then define $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$. Then,

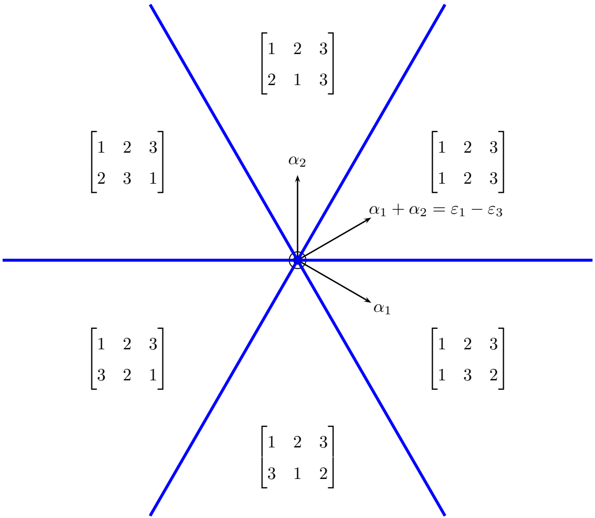
$$s_{\alpha_1}(\varepsilon_1) = \varepsilon_1 - \frac{2\langle \varepsilon_1, \varepsilon_1 - \varepsilon_2 \rangle}{\langle \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2 \rangle}(\varepsilon_1 - \varepsilon_2) = \varepsilon_1 - (\varepsilon_1 - \varepsilon_2) = \varepsilon_2,$$

$$s_{\alpha_1}(\varepsilon_2) = \varepsilon_2 - \frac{2\langle \varepsilon_2, \varepsilon_1 - \varepsilon_2 \rangle}{\langle \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2 \rangle}(\varepsilon_1 - \varepsilon_2) = \varepsilon_2 + (\varepsilon_1 - \varepsilon_2) = \varepsilon_1.$$

so,

$$s_{\alpha_1} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ \varepsilon_2 & \varepsilon_1 & \varepsilon_3 \end{bmatrix},$$

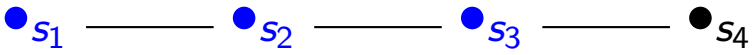
$$s_{\alpha_2} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ \varepsilon_1 & \varepsilon_3 & \varepsilon_2 \end{bmatrix}.$$


1 2 3
2 1 3
1 2 3
2 3 1
1 2 3
1 2 3
α_2
$\alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_3$
α_1
1 2 3
3 2 1
1 2 3
1 3 2
1 2 3
3 1 2

Symmetric Group

A *subgroup* of S_n is any subset of permutations that is closed under the composition operation.

One special way this can happen is by taking a **subset of the generators**, called a *parabolic subgroup*.
For example, S_4 is a parabolic subgroup of S_5:



These are all the permutations in which the last entry 5 **is fixed**.

$$S_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ * & * & * & * & 5 \end{bmatrix}$$

Symmetric Group

If we wanted to use our understanding of S_4 to understand S_5, we could specify permutations in S_5 by

(1) Permute the first four entries.

(2) Move the entry 5 into its final position.

For example, we could build $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{bmatrix}$ as

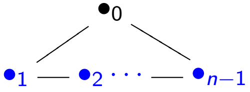
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \xrightarrow{s_1} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{bmatrix} \xrightarrow{s_3} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{bmatrix}$$

$$\xrightarrow{s_4} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{bmatrix} \xrightarrow{s_3} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{bmatrix}$$

Affine Symmetric Group

The *affine symmetric group* $\widetilde{S_n}$ is presented as a Coxeter group by:

- Generators $s_0, s_1, \ldots, s_{n-1}$, with $s_i^2 = id$,
- Commuting relations $s_i s_j = id$ if $|i - j| \geq 2$,
- Braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_0 s_{n-1} s_0 = s_{n-1} s_0 s_{n-1}$.



This is an **infinite** Coxeter group, but notice that it has **finite** S_n as a **parabolic subgroup**.

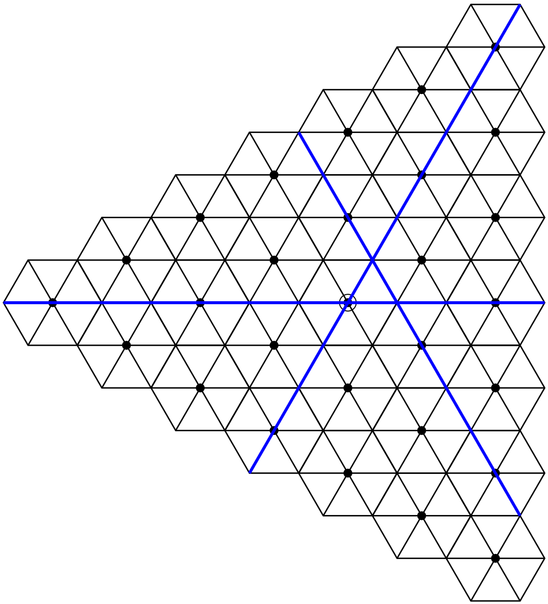
Affine Symmetric Group

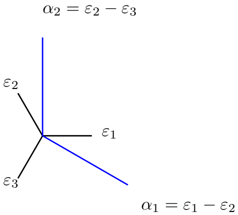
We can again view the generators s_i as **reflections** of a vector space with orthonormal basis $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$. The formula for reflection by $s_1, s_2, \ldots, s_{n-1}$ is exactly the same as before:

- For $1 \leq i \leq n-1$, let s_i be the reflection that interchanges ε_i and ε_{i+1}

The reflection by s_0 is an **affine reflection** defined on $v = \sum_{j=1}^{n} a_j \varepsilon_j$ by

$$s_0(v) = (a_n + 1)\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_{n-1}\varepsilon_{n-1} + (a_1 - 1)\varepsilon_n.$$





$\alpha_2 = \varepsilon_2 - \varepsilon_3$
ε_2
ε_1
ε_3
$\alpha_1 = \varepsilon_1 - \varepsilon_2$

Affine Symmetric Group

The *simple roots* Δ *of type* A_{n-1} are

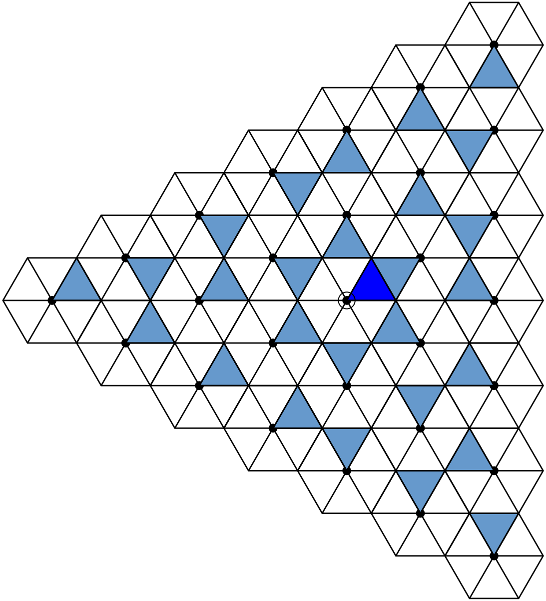
$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \ldots, \quad \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n.$$

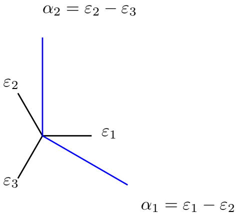
The $\mathbf{Z}$-span Λ_R of Δ is called the *root lattice of type* A_{n-1}. Note that these are just **vectors whose coordinates sum to 0**.

Recall that S_n is a **parabolic subgroup** of $\widetilde{S_n}$. It turns out that there is a unique way to write any affine permutation as a pair

(element of the root lattice , finite permutation)

However, it's better to look at all the affine permutations that correspond to a given root lattice point, and choose a special one to represent the root lattice point. This affine permutation called a *minimal length coset representative*.





$\alpha_2 = \varepsilon_2 - \varepsilon_3$
ε_2
ε_1
ε_3
$\alpha_1 = \varepsilon_1 - \varepsilon_2$

Affine Symmetric Group

We want to study

minimal length coset representatives

$\leftrightarrow$ integer vectors whose coordinates sum to 0

$\leftrightarrow$ *n*-cores

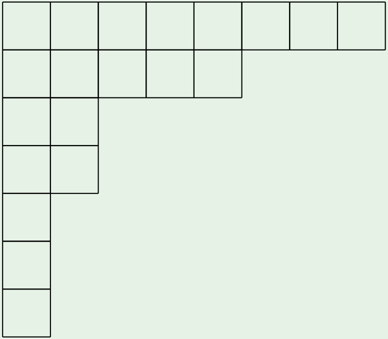
$\leftrightarrow$ abacus diagrams

and especially, **how to project an n-core to an $(n-1)$-core**?
E.g.

Core Notation

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$ be a *partition* and $n \geq 2$ be an integer.

Example



The *n-residue* of a box (i, j) is the least nonnegative integer $\equiv j - i$ mod n.

Core Notation

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$ be a *partition* and $n \geq 2$ be an integer.

Example

0	1	2	3	0	1	2	3
3	0	1	2	3			
2	3						
1	2						
0							
3							
2							

The *n-residue* of a box (i, j) is the least nonnegative integer $\equiv j - i \mod n$.

Core Notation

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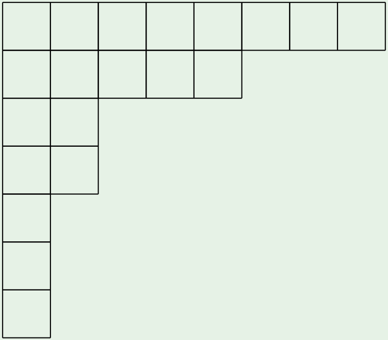
Example



The *hook length* of a box (i, j) is the number of boxes to the **right and below** the box, including itself. It is denoted $h^{\lambda}_{(i,j)}$.

Core Notation

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$ be a *partition* and $n \geq 2$ be an integer.

Example

14	10	7	6	5	3	2	1
10	6	3	2	1			
6	2						
5	1						
3							
2							
1							

The *hook length* of a box (i, j) is the number of boxes to the **right and below** the box, including itself. It is denoted $h^{\lambda}_{(i,j)}$.

Cores

Definition

A partition λ is an *n-core* if $n \nmid h^{\lambda}_{(i,j)}$ for every box (i,j) of λ.

Example

14	10	7	6	5	3	2	1
10	6	3	2	1			
6	2						
5	1						
3							
2							
1							

λ is a 4-core.

Cores

Question

Given an n-core, how can we project to obtain an $(n-1)$-core?

Example

9	**5**	**3**	**1**
7	3	1	
5	**1**		
3			
2			
1			

4-core

$\longrightarrow$

5	**2**	**1**
2		
1		

3-core

$\longrightarrow$

1

2-core

Cores

n-core partitions index:

- Schubert cells in the affine Grassmannian Gr of $SL(n, \mathbb{C})$.
  $(\mathrm{Gr} \cong SL_n(\mathbb{C}((t)))/SL_n(\mathbb{C}[[t]]).)$
- k-Schur functions and dual k-Schur functions in $H_*(\mathrm{Gr}) \cong \Lambda_n$ and $H^*(\mathrm{Gr}) \cong \Lambda^n$, respectively.
- Blocks in the representation theory of the symmetric group S_k over a field of characteristic $n > 0$.

Cores

- $\mathcal{C}_n$ = The set of all n-cores.
- $\mathcal{C}_n^k$ = The subset of $\mathcal{C}_n$ having first part k.
- $\mathcal{C}_{n-1}^{\leq k}$ = The subset of $\mathcal{C}_{n-1}$ having first part $\leq k$.

We will define a bijection

$$\Phi_n^k : \mathcal{C}_n^k \to \mathcal{C}_{n-1}^{\leq k}$$

Then,

$$\sum_{k \geq 0} |C_n^k| x^k = \sum_{k \geq 0} \binom{k+n-2}{k} x^k = \frac{1}{(1-x)^{n-1}}.$$

(Proof:

$$\binom{k+n-2}{k} = \binom{k+n-3}{k} + \binom{k+n-4}{k-1} + \cdots + \binom{n-3}{0}.)$$

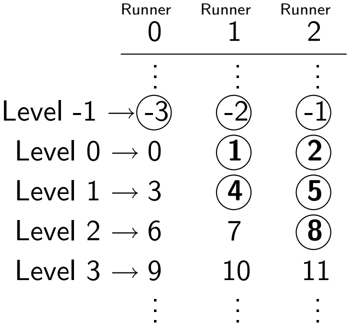
Beta numbers and Abaci

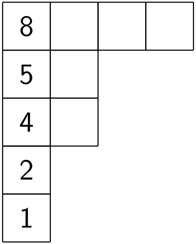
The partition shape is determined by first column hooklengths. These can be generalized to β-numbers.

8			
5			
4			
2			
1			

⋯	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	⋯
⋯	○	○	○	●	○	○	●	○	○	●	●	○	●	●	●	⋯

Beta numbers and Abaci



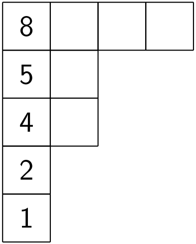


3-core

⋯	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	⋯
⋯	○	○	○	●	○	○	●	○	○	●	●	○	●	●	●	⋯

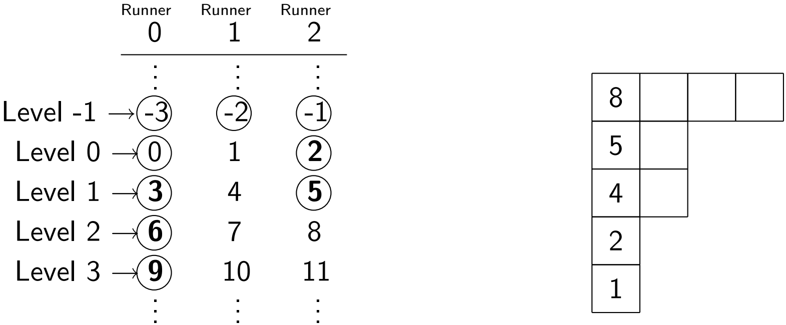
Beta numbers and Abaci

	Runner 0	Runner 1	Runner 2
	⋮	⋮	⋮
Level -1 →	-3	-2	-1
Level 0 →	0	**1**	**2**
Level 1 →	3	**4**	**5**
Level 2 →	6	7	**8**
Level 3 →	9	10	11
	⋮	⋮	⋮



The abacus for $\beta = (\mathbf{8}, \mathbf{5}, \mathbf{4}, \mathbf{2}, \mathbf{1}, -1, -2, -3, \ldots)$ has balance number $2 = (-1) + 1 + 2$.

Beta numbers and Abaci



The abacus for $\beta = (\mathbf{8}, \mathbf{5}, \mathbf{4}, \mathbf{2}, \mathbf{1}, -1, -2, -3, \ldots)$ has balance number 2.
The abacus for $\beta = (9, 6, 5, 3, 2, 0, -1, -2, \ldots)$ has balance number $3 = 3 + (-1) + 1$.

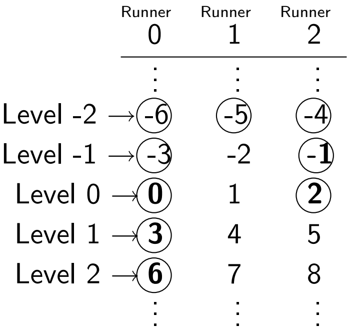
Beta numbers and Abaci

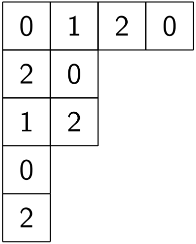
Theorem

Theorem 2.7.16, Lemma 2.7.38 in James–Kerber

- *λ is an n-core if and only if any (equivalently, every) abacus of λ on n runners is flush.*
- *Moreover, in the* **balanced flush abacus** *of an n-core λ, each active bead on runner i corresponds to a row of λ whose rightmost box has residue i.*

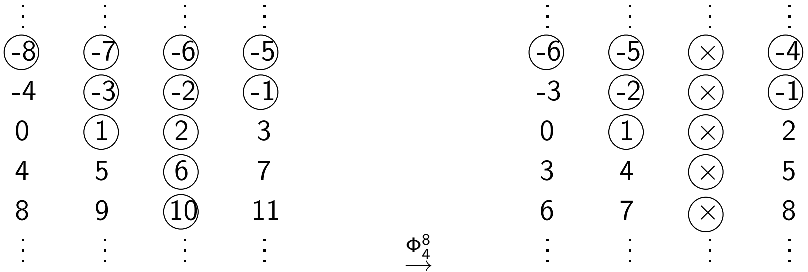
Beta numbers and Abaci





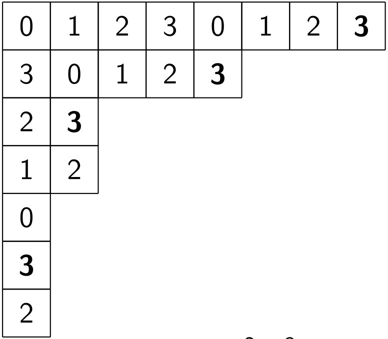
3-core

The bijection Φ_n^k



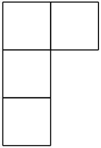
4-core $(8, 5, 2^2, 1^3)$ $\xrightarrow{\Phi_4^8}$ 3-core $(2, 1^2)$.

The bijection Φ_n^k



4-core $(8, 5, 2^2, 1^3)$

$\xrightarrow{\Phi_4^8}$



3-core $(2, 1^2)$.

The bijection Φ_n^k

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \Lambda_R$ written in the ε_i basis, so each $a_i \in \mathbf{Z}$ and $\sum_{i=1}^n a_i = 0$.

We form a balanced flush abacus from $\mathbf{a}$ by filling the $(i-1)^{\mathrm{st}}$ runner with beads from $-\infty$ down to level a_i.

This defines a bijection

$$\pi : \{(a_1, \ldots, a_n) : a_i \in \mathbf{Z}, \sum_{i=1}^{n} a_i = 0\} \to \{\text{balanced flush abaci}\} \to \mathcal{C}_n.$$

The bijection Φ_n^k

Example

$n = 4$, $(2, 0, 0, -2)$ cooresponds to

-8	-7	-6	-5
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8	9	10	11

0	1	2	3	**0**
3	**0**			
2				
1				
0				

The bijection Φ_n^k

Proposition

Suppose that $\pi(\mathbf{a}) = \pi(a_1, \ldots, a_n) = \lambda$. *Then we have*

$$\lambda_1 = (a_i - 1)n + i$$

where a_i *is the rightmost occurrence of the largest coordinate in* $\mathbf{a}$.

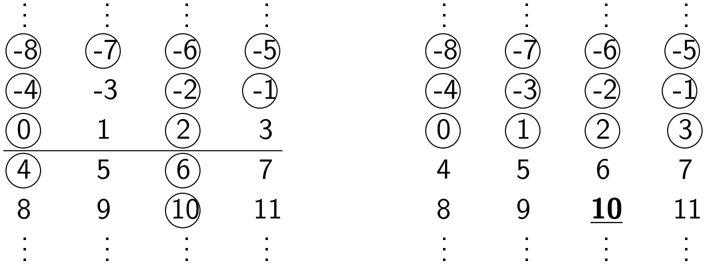
Corollary

For $k \geq 0$, *let* H_n^k *denote the affine hyperplane*

$$H_n^k = \{\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{R}^n : (\mathbf{a}, \varepsilon_{(k \mod n)}) = \lceil \frac{k}{n} \rceil\} \cap V$$

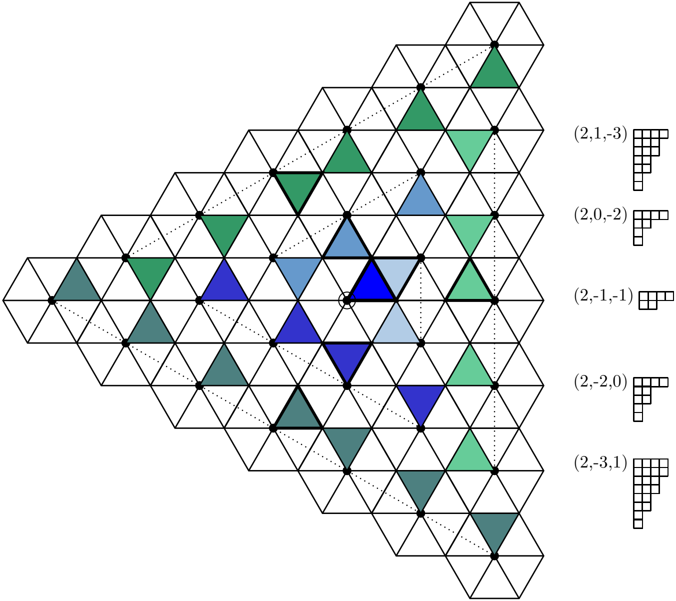
inside V, *where* $1 \leq (k \mod n) \leq n$. *Then under the correspondence* π, *the* n*-cores* λ *with* $\lambda_1 = k$ *all lie inside* $H_n^k \bigcap \Lambda_R$.

The bijection Φ_n^k



$$7 = \lambda_1 = (a_i - 1)n + i = (2-1)4 + 3.$$

$$H_4^7 = \{(a_1, a_2, a_3, a_4) : a_3 = 2\} \cap V$$


(2,1,-3)
(2,0,-2)
(2,-1,-1)
(2,-2,0)
(2,-3,1)

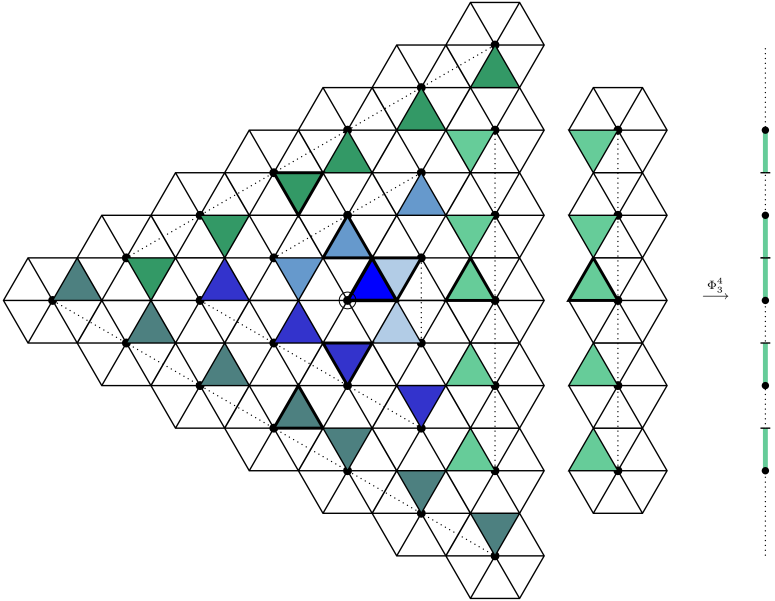
The bijection Φ_n^k

Theorem

Let ψ_n be the affine map defined by $\psi_n(a_1, \ldots, a_n) = (a_n + 1, a_1, a_2, \ldots, a_{n-1})$. Then,

$$\pi^{-1} \circ \Phi_n^k \circ \pi(a_1, \ldots, a_n) = \psi_{n-1}^{a_i}(a_1, \ldots, \widehat{a_i}, \ldots, a_n)$$

where a_i is the rightmost occurrence of the largest entry among $\{a_1, \ldots, a_n\}$ and the circumflex indicates omission.


$\xrightarrow{\Phi_3^4}$

The bijection Φ_n^k

We can factor this map into **translation** composed with **root system embedding**.

Example

Let $n = 3$. The affine hyperplane H_3^7 contains the partition $\pi(3, 1, -4) = (7, 5, 4^2, 3^2, 2^2, 1^2)$. Translation by $\mathbf{t} = (-3, 1, 2)$ sends H_3^7 to

$$\{(a_1, a_2, a_3) \in V : a_1 = 0\}$$

and in particular sends $(3, 1, -4)$ to $(0, 2, -2)$.

We view this as a subspace of $\mathbf{R}^2$ with orthonormal basis $\{e_1', e_2'\}$ and A_{n-2} root system. The embedding identifies e_1' with e_3 and e_2' with e_2 and we have $\psi^3(1, -4) = (-2, 2)$ corresponding to $\Phi_3^7(7, 5, 4^2, 3^2, 2^2, 1^2) = (4, 3, 2, 1)$.

Open questions:

- How do these combinatorics generalize to other reflection groups?
- What does the projection Φ_n^k imply about cells in the affine Grassmannian, k-Schur functions, or blocks in S_n-modules?