Tropical Geometry

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The **tropical semiring** $(\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$ has:

- •"addition" $\oplus$ usual minimum.
- •"multiplication" $\otimes$ usual addition

Examples:

$$3 \oplus 4 = 3, \quad 5 \otimes 7 = 12.$$

Tropical arithmetic is associative and distributative:

$$(3 \oplus 7) \otimes 5 = (3 \otimes 5) \oplus (7 \otimes 5) = 8$$

"Freshman's dream": $(a \oplus b)^7 = a^7 \oplus b^7$.

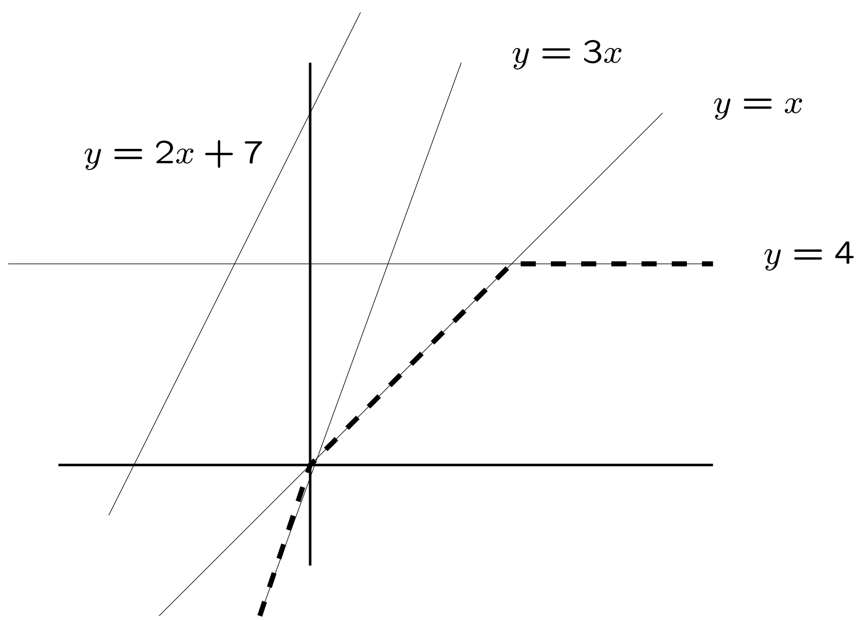
The additive identity is ∞, and the multiplicative identity is 0.

Warning: No additive inverses!

Tropical polynomials are piecewise linear functions:

Example: $F = x^3 \oplus 7 \otimes x^2 \oplus x \oplus 4$

$$F(x) = \min(3x, 2x+7, x, 4)$$



Problem: With no additive inverse, what does $3 \otimes x \oplus -2 = 0$ mean?

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Answer: "Solutions" to such equations are points of nondifferentiability of the graph of F.

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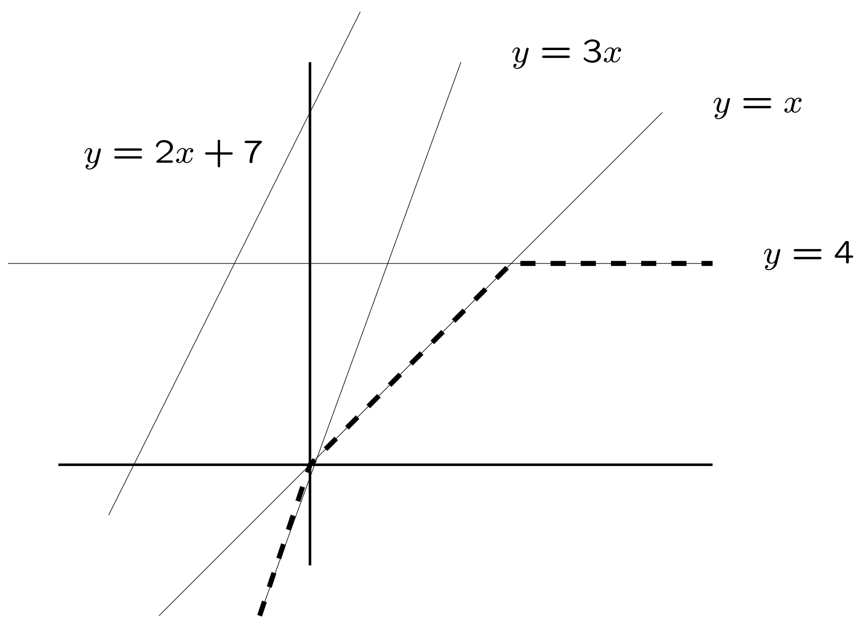
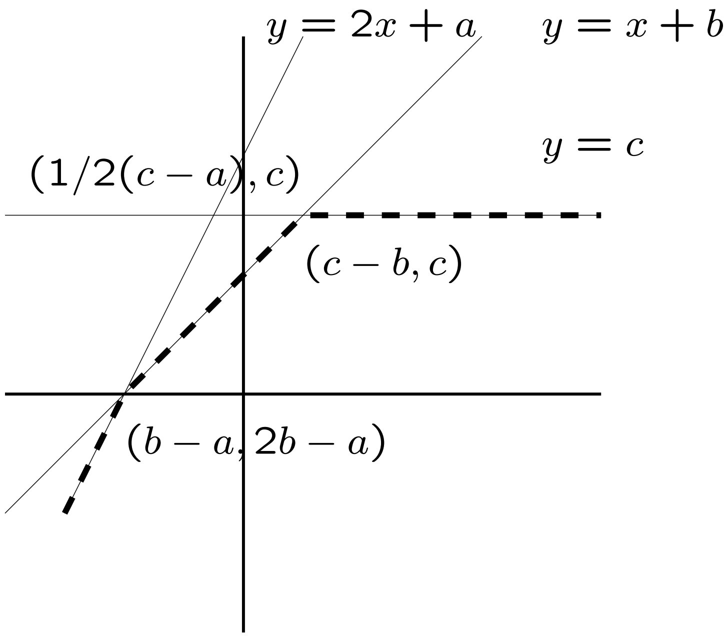
Answer: "Solutions" to such equations are points of nondifferentiability of the graph of F.

Example: $F = x^3 \oplus 7 \otimes x^2 \oplus x \oplus 4 = 0$ for $x = 0, 4$.



Tropical quadratic formula: $F = a \otimes x^2 \oplus b \otimes x \oplus c = \min(2x + a, x + b, c)$.

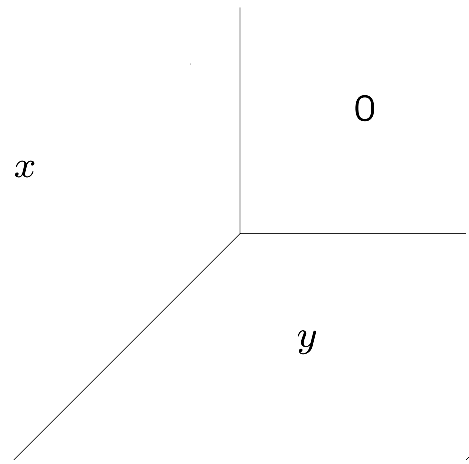


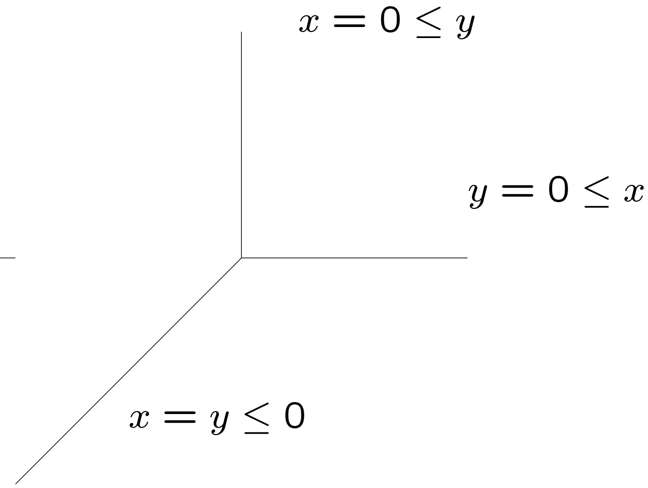
"Solutions":

$$x = \begin{cases} b-a, c-b & \text{if } 2b \le a+c \\ 1/2(c-a) & \text{if } 2b > a+c \end{cases}$$

Definition: The **tropical hypersurface** $V(F)$ defined by the tropical polynomial F is the nondifferentiability locus of the graph of F.

Example: $F = x \oplus y \oplus 0 \qquad F(x) = \min(x, y, 0)$.





$w \in V(F)$ if and only if the **minimum is achieved at least twice** in $F(w)$.

Definition: A (affine complex) variety is the common solutions of a set of polynomial equations

$$X = V(f_1, \ldots, f_s) = \{x \in \mathbb{C}^n : f_1(x) = \cdots = f_s(x) = 0\}.$$

Example

$$\begin{aligned} X =& V(x + y + 1, x + 2y + 3z) \\ =& \{(x, y, z) \in \mathbb{C}^3 : x + y = -1, x + 2y + 3z = 0\} \\ =& \{(3t - 2, 1 - 3t, t) : t \in \mathbb{C}\} \end{aligned}$$

Definition: A (**very** affine complex) variety is the common **nonzero** solutions of a set of (**Laurent**) polynomial equations.

$$X = V(f_1, \ldots, f_s) = \{x \in (\mathbb{C}^*)^n : f_1(x) = \cdots = f_s(x) = 0\}.$$

Example

$$\begin{aligned} X =& V(x + y + 1, x + 2y + 3z) \\ =& \{(x, y, z) \in (\mathbb{C}^*)^3 : x + y = -1, x + 2y + 3z = 0\} \\ =& \{(3t - 2, 1 - 3t, t) : t \in \mathbb{C} \setminus \{0, 1/3, 2/3\}\} \end{aligned}$$

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Note: X only depends on the variety defined by the **ideal** generated by the Laurent polynomials $f_1, \ldots, f_s$:

$$I = \langle f_1, \ldots, f_s \rangle = \left\{ \sum_{i=1}^{s} g_i f_i : g_i \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \right\}.$$

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If $\langle f_1, \ldots, f_s \rangle = \langle f_1', \ldots, f_r' \rangle$ then $V(f_1, \ldots, f_s) = V(f_1', \ldots, f_r')$.

Write $X = V(I)$ for $I = \langle f_1, \ldots, f_s \rangle$, and $I = I(X)$.

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If $\langle f_1, \ldots, f_s \rangle = \langle f'_1, \ldots, f'_r \rangle$ then $V(f_1, \ldots, f_s) = V(f'_1, \ldots, f'_r)$.

Write $X = V(I)$ for $I = \langle f_1, \ldots, f_s \rangle$, and $I = I(X)$.

Example: $X = V(x + y + 1, x + 2y + 3z)$.

$I = \langle x + y + 1, x + 2y + 3z \rangle = \langle y + 3z - 1, x - 3z + 2 \rangle$

Notation: For $u \in \mathbb{Z}^n$, $x^{\mathbf{u}} = x_1^{u_1} x_2^{u_2} \ldots x_n^{u_n}$.

Definition: If $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a Laurent polynomial, then

$$F = \mathrm{trop}(f) := \bigoplus x^u = \min_{c_u \neq 0} w \cdot u$$

The tropical hypersurface $\mathtt{trop}(V(f))$ of f is the tropical hypersurface of F.

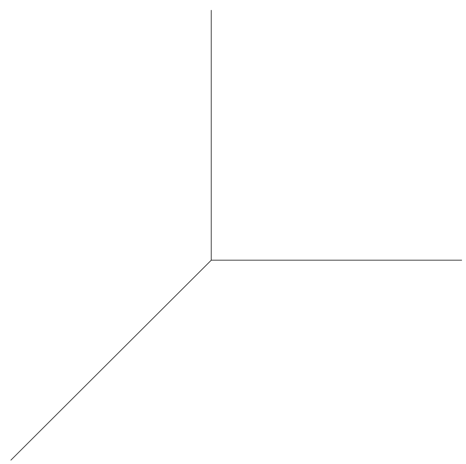
(**Warning**: Coefficients have disappeared!)

Example: Let $f = x + y + 1$. Then $\mathrm{trop}(f) = x \oplus y \oplus 0 = \min(x, y, 0)$.

Definition: Let $X = V(I) \subset (\mathbb{C}^*)^n$. The **tropical variety** of X is

$$\mathtt{trop}(X) = \bigcap_{f \in I(X)} \mathtt{trop}(V(f))$$

Theorem [Kapranov, Speyer, Sturmfels, Bieri/Groves ...] The tropical variety of X is the support of a balanced polyhedral fan of the same dimension as X that is connected in codimension-one.



Why?

Tropical varieties are **combinatorial shadows** of classical varieties

Many invariants of the variety can be determined from
the combinatorics of the tropical variety.

Algebraic Geometry (hard) $\rightsquigarrow$ **Polyhedral Geometry/Combinatorics**

(somewhat easier)

Example: Let $X = V(x+y+1, x+2y+3z)$.

Then $\text{trop}(V(x+y+1)) = \{(w_1, w_2, w_3) : w_1 = w_2 \leq 0$ or $w_1 = 0 \leq w_2$ or $w_2 = 0 \leq w_1\}$

$\text{trop}(V(x+2y+3z)) = \{(w_1, w_2, w_3) : w_1 = w_2 \leq w_3$ or $w_1 = w_3 \leq w_2$ or $w_2 = w_3 \leq w_1\}$

So $(-1,-1,1) \in \text{trop}(V(x+y+1)) \cap \text{trop}(V(x+2y+3z))$. However $y+3z-1 \in I$, and $(-1,-1,1) \notin \text{trop}(V(y+3z-1))$.

Example continued: $X = V(x + y + 1, x + 2y + 3z)$.

$\text{trop}(X) \subsetneq \text{trop}(V(x + y + 1)) \cap \text{trop}(V(x + 2y + 3z))$

In fact $\text{trop}(X) = \{(w_1, w_2, w_3) : w_1 = w_2 = w_3 \leq 0$ or $w_1 = w_2 = 0 \leq w_3, w_1 = w_3 = 0 \leq w_2$ or $w_2 = w_3 = 0 \leq w_1\}$.

This is a fan with rays spanned by $\{(-1, -1, -1), (0, 0, 1), (0, 1, 0), (1, 0, 0)\}$.

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Question: Can we write $\text{trop}(X)$ as a **finite** intersection of $\text{trop}(V(f))$ for $f \in I(X)$?

Linear varieties

If $X = V(f_1, \ldots, f_r) \subset (\mathbb{C}^*)^n$ where $f_i = \sum_{j=1}^{n} a_{ij}x_j + b_i$, then $X = \{x \in (\mathbb{C}^*)^n : Ax = -b\}$, where A is the $r \times n$ matrix $A = (a_{ij})$, and $b = (b_i) \in \mathbb{C}^n$.

$x \in X$ if and only if $(x, 1) \in (\mathbb{C}^*)^{n+1} \cap \ker(A|b)$.

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Example: $X = V(x + y + 1, x + 2y + 3z)$.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}.$$

$\mathbf{b} = (1, 0)$

Generators for $I(X)$ correspond to bases for the row space of $(A|b)$.

Definition: The support of $v \in \mathbb{C}^{n+1}$ is $\{i : v_i \neq 0, 1 \leq i \leq n+1\}$.

A **circuit** of the matrix $(A|b)$ is a vector $v \in \mathbb{C}^{n+1}$ in the row space of $(A|b)$ whose support is minimal with respect to inclusion.

Note: There are finitely many circuits up to scaling.

Example:

$$(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{array} \right).$$

Circuits: $\{(1,1,0,1),(1,2,3,0),(0,1,3,-1),(1,0,-3,2)\}$.

Theorem: Let $X = \{x \in (\mathbb{C}^*)^n : Ax = -b\}$ where A is an $r \times n$ matrix and $b \in \mathbb{C}^r$. Let $\mathcal{C} = \{\sum_{i=1}^n c_i x_i + c_{n+1} : c = (c_1, \ldots, c_{n+1})$ is a circuit of $(A|b)\}$. Then

$$\mathrm{trop}(X) = \bigcap_{f \in \mathcal{C}} \mathrm{trop}(V(f))$$

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$$\mathrm{trop}(X) = \bigcap_{f \in \mathcal{C}} \mathrm{trop}(V(f))$$

Example: $\mathrm{trop}(V(x+y+1, x+2y+3z)) = \mathrm{trop}(V(x+y+1)) \cap \mathrm{trop}(V(x+2y+3z)) \cap \mathrm{trop}(V(y+3z-1)) \cap \mathrm{trop}(V(x-3y+2))$.

This can be computed using the program **gfan** by Anders Jensen.

Open question: Give as nice an answer for general (nonlinear) varieties.

Why linear varieties?

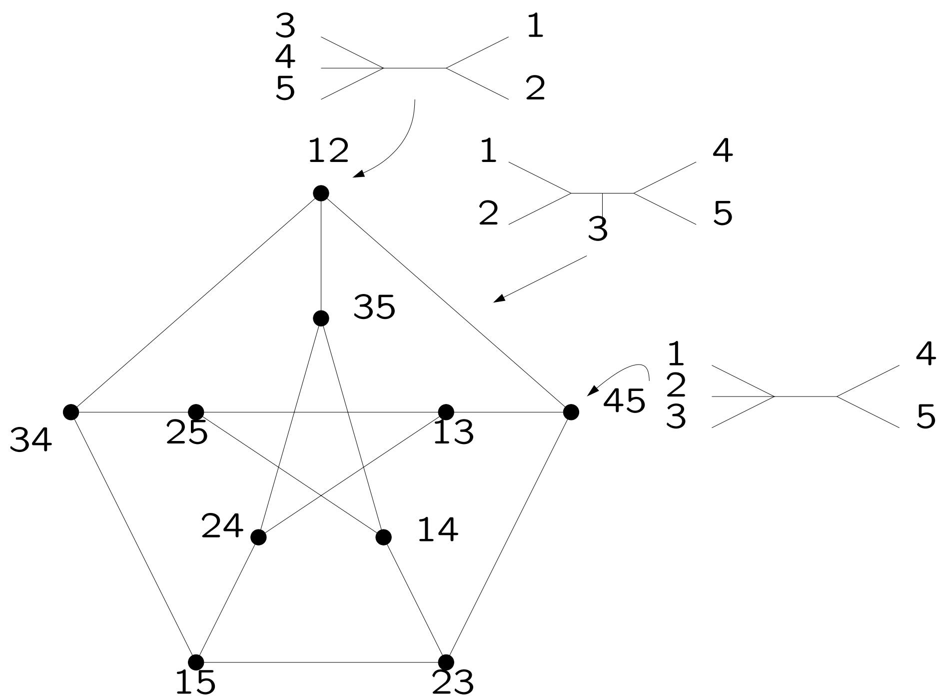
Many interesting varieties in $(\mathbb{C}^*)^n$ are cut out by linear equations.

Example: $X = M_{0,n}$, the moduli space of n distinct points on $\mathbb{P}^1$.

$$\begin{aligned} M_{0,n} &= (\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus \text{ diagonals} \\ &= (\mathbb{C}^* \setminus \{1\})^{n-3} \setminus \text{ diagonals} \end{aligned}$$

There is an embedding $\phi : M_{0,n} \to (\mathbb{C}^*)^{\binom{n}{2}-n}$ for which $\phi(M_{0,n})$ is cut out by linear equations.

$\Delta = \mathrm{trop}(M_{0,n})$ is the well-studied **space of phylogenetic trees**.



The fan Δ determines a **toric variety** X_Δ, which is a (partial) compactification of $(\mathbb{C}^*)^{\binom{n}{2}-n}$. The closure of $M_{0,n}$ inside X_Δ is equal to $\overline{M}_{0,n}$, the moduli space of stable genus zero curves with n marked points (the Deligne-Mumford compactification of $M_{0,n}$)

Open question: How much of the geometry of $\overline{M}_{0,n}$ can be determined from the combinatorics of Δ? (on going joint work with Angela Gibney).

Other applications:

- Enumerative geometry (Mikhalkin, Gathmann-Markwig, . . .)
- Arithmetic geometry (Gubler, Baker, . . .)
- Real algebraic geometry (Itenberg, Shustin, . . .)
- Compactifications of moduli spaces (Tevelev, Keel, Hacking, . . . ,)
- Combinatorics (Develin, Ardila, . . .)

Many basics foundational issues remain (Sturmfels, Speyer, Payne, . . .)

Mathematical Sciences Research Institute

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Introductory workshop: August 24-28, 2009.