Arithmetic of arithmetic Coxeter groups

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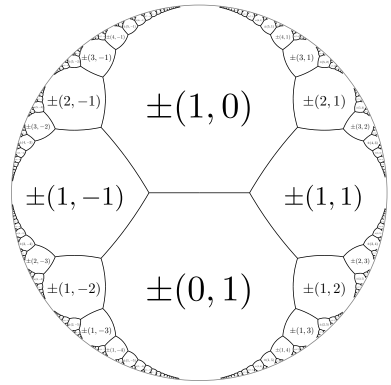
BQFs over $\mathbb{Z}$ and Conway's topograph

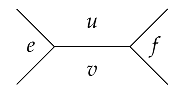


A *binary quadratic form* is a function $Q\colon \mathbb{Z}^2 \to \mathbb{Z}$, of the form

$$Q(x,y) = ax^2 + bxy + cy^2$$

with $a,b,c \in \mathbb{Z}$. Its discriminant is $\Delta = b^2 - 4ac$.

The range topograph of Q is obtained by replacing each primitive lax vector $\pm(x,y)$ by $Q(x,y)$ in the margin-figure. The geometry of the topograph comes from an isomorphism, $PGL_2(\mathbb{Z}) \cong (3,\infty)$.

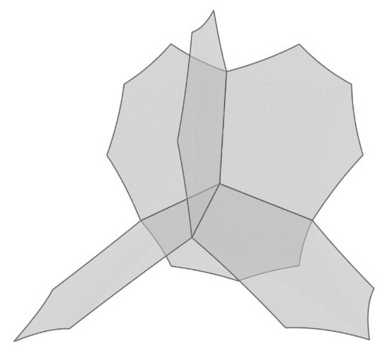


Arith progression: e, $u+v$, f

At every cell in the topograph (pictured in the margin), the values form an arithmetic progression: $(u+v) - e = f - (u+v)$. Also, $\Delta = (u-v)^2 - ef$.

Let μ_Q be the minimum nonzero absolute value of Q. If Δ is a positive nonsquare, then Q is an indefinite form and $\mu_Q \leq \sqrt{\Delta/5}$.

BHFs over $\mathbb{E} = \mathbb{Z}[e^{2\pi i/3}]$



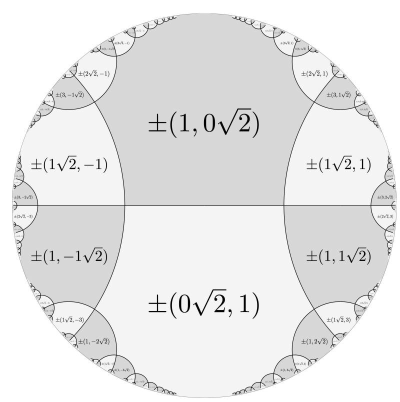
A *binary Hermitian form* is a function $H\colon \mathbb{E}^2 \to \mathbb{E}$ of the form

$$Q(z,w) = az\bar{z} + bz\bar{w} + \bar{b}\bar{z}w + cw\bar{w}$$

with $a,c \in \mathbb{Z}$ and $b \in \mathbb{E}$ (or the inverse different).

The geometry of the topograph captures lax vectors, bases, superbases, and "tetrabases," and comes from an isomorphism $PSL_2(\mathbb{E}) \cong (3,3,6)^+$. It is composed of hexagons, meeting with tetrahedral symmetry at each vertex.

BQDs over $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{3}]$



Let $\sigma = 2$ or $\sigma = 3$ and $R = \mathbb{Z}[\sqrt{\sigma}]$. The *dilinear group* $DL_2(R)$ is the group of invertible matrices with entries in R, where one diagonal has entries in $\mathbb{Z}$ and the other diagonal has entries in $\mathbb{Z} \cdot \sqrt{\sigma}$.

Red divectors are vectors of the form (x,y) with $x \in \mathbb{Z}$ and $y \in \mathbb{Z} \cdot \sqrt{\sigma}$. *Blue divectors* have $y \in \mathbb{Z}$ and $x \in \mathbb{Z} \cdot \sqrt{\sigma}$. A *binary quadratic diform* is a function, $Q\colon \{ \text{ divectors } \} \to \mathbb{Z}$ of the form

$$Q(x,y) = ax^2 + b\sqrt{\sigma}xy + cy^2,$$

with $a,b,c \in \mathbb{Z}$. The discriminant is $\Delta = \sigma^2 b^2 - 4\sigma ac$. Restricting Q to red/blue divectors yields two "linked" BQFs of discriminant Δ.

The topograph comes from the isomorphism $PDL_2(R) \cong (2\sigma, \infty)$.